

The dihedral Lie algebras and Galois symmetries of $\pi_1^{(l)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$

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1 An outline

This is the next in the series of papers [G1-3] devoted to study of higher cyclo-
tomy, understood as motivic theory of multiple polylogarithms at roots of unity,
and its relationship with modular varieties for GL_m/\mathbb{Q} , for all $m \geq 1$. Let μ_N
be the group of N -th roots of unity. Our main objective is a mysterious link
between the *structure of the motivic fundamental group of*

$$X_N := \mathbb{P}^1 - (\{0, \infty\} \cup \mu_N) = \mathbb{G}_m - \mu_N$$

and the *geometry and topology of the following modular varieties for GL_m/\mathbb{Q} :*

$$\Gamma_1(m; N) \backslash GL_m(\mathbb{R})/\mathbb{R}_+^* \cdot O_m \quad \text{for } m > 1 \quad (1)$$

where $\Gamma_1(m; N) \subset GL_m(\mathbb{Z})$ is the subgroup stabilizing the vector $(0, \dots, 0, 1)$
mod N . For $m = 1$ it is $S_N := \text{Spec}(\mathbb{Z}[\zeta_N][\frac{1}{N}])$. Adelic approach provides a
coherent description for all m , see s. 2.2.

In the present paper we turn to the Galois side of the story. However to keep the motivic perspective let us recall the following. According to Deligne [D] the motivic fundamental group of a variety is not just a group, but rather a Lie algebra object in the category of mixed motives. It can be viewed as a pronilpotent completion of the topological fundamental group of the corresponding complex variety equipped with lots of additional structures of analytic, geometric and arithmetic nature.

Mixed motives are algebraic geometric objects. They can be seen through their realizations. The two most popular realizations which hypothetically capture all the information about the category of the mixed motives are:

- the Hodge realization* provided by analysis and algebraic geometry, and
- the l -adic realization* provided by arithmetic and algebraic geometry.

The l -adic realization of the motivic fundamental group is obtained from the action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the pro- l completion $\pi_1^{(l)}(X)$ of the fundamental group of the variety X . That is why in this paper we study the action of the Galois group on $\pi_1^{(l)}(X_N)$.

In the case $N = 1$ this problem has been addressed by Grothendieck [Gr], Deligne [D], Ihara [Ih0-3], Drinfeld [D] and others - see the wonderful survey of Ihara [Ih0]. For $N > 1$ it has not been investigated. Our point of view is that we should study the problem for all N , penetrating the structures independent of N (like modular complexes, see s. 2.5).

Briefly our approach is this. Let G be a commutative group. In [G3] we constructed a bigraded Lie algebra $D_{\bullet\bullet}(G)$, called the dihedral Lie algebra of G , see section 3 below. We relate $D_{\bullet\bullet}(\mu_N)$ to the Lie algebra of the image of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\text{Aut}\pi_1^{(l)}(X_N)$. On the other hand in [G2-3, 5] the structure of the Lie algebra $D_{\bullet\bullet}(\mu_N)$ is related to the geometry of modular varieties (1). Using these results we study the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{(l)}(X_N)$. In particular we obtain new results about the action on $\pi_1^{(l)}(\mathbb{P}^1 - \{0, 1, \infty\})$.

Here is a more detailed account. Let X be a regular curve over $\overline{\mathbb{Q}}$. Let \overline{X} be the corresponding projective curve and v_x a nonzero tangent vector at a point $x \in \overline{X}$. Then according to Deligne [D] one can define the geometric profinite fundamental group $\widehat{\pi}_1(X, v_x)$ based at the vector v_x . If X , x and v_x are defined over a number field $F \subset \overline{\mathbb{Q}}$ then the group $\text{Gal}(\overline{\mathbb{Q}}/F)$ acts by automorphisms of $\widehat{\pi}_1^{(l)}(X, v_x)$. If $X = X_N$, there is a tangent vector v_∞ at ∞ corresponding to the inverse t^{-1} of the canonical coordinate t on $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N)$.

Since any finite l -group is nilpotent the pro- l group $\pi_1^{(l)}(X_N, v_\infty)$ is pronilpotent. Let $\mathbb{L}_N^{(l)} = \mathbb{L}^{(l)}(X_N, v_\infty)$ be the l -adic pro-Lie algebra corresponding via the Maltsev theory to $\pi_1^{(l)}(X_N, v_\infty)$ (see ch. 9 of [D]). It is a free pronilpotent Lie algebra with generators corresponding to the loops around 0 and all N -th roots of unity. We call it the l -adic fundamental Lie algebra of X_N . The Galois group acts by its automorphisms.

Let ζ_n be a primitive n -th root of unity, and $\mathbb{Q}(\zeta_{l^\infty N}) := \cup \mathbb{Q}(\zeta_{l^n N})$. For

the reasons explained in s 3.1 we restrict the action of the Galois group to the subgroup $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$, picking up the homomorphism

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) \longrightarrow \text{Aut} \mathbb{L}_N^{(l)}$$

Let $\text{Der} \mathbb{L}_N^{(l)}$ be the Lie algebra of all derivations of the Lie algebra $\mathbb{L}_N^{(l)}$. Linearizing, as explained in s. 2.1 or 3.2, the above map we get the Lie algebra

$$\mathcal{G}_N^{(l)} \hookrightarrow \text{Der} \mathbb{L}_N^{(l)} \quad (2)$$

The fundamental Lie algebra $\mathbb{L}_N^{(l)}$ is equipped with two filtrations preserved by the Galois action. The weight filtration can be defined on the fundamental Lie algebra of any algebraic variety over \mathbb{Q} . In our case it coincides with the lower central series of the Lie algebra $\mathbb{L}_N^{(l)}$. The depth filtration is more specific. It is given by the lower central series of the codimension one ideal

$$\mathcal{I}_N := \text{Ker} \left(\mathbb{L}^{(l)}(X_N, v_\infty) \longrightarrow \mathbb{L}^{(l)}(\mathbb{G}_m, v_\infty) = \mathbb{Q}_l(1) \right)$$

where the map is provided by the natural inclusion $X_N \hookrightarrow \mathbb{G}_m$.

These filtrations induce filtrations on $\text{Der} \mathbb{L}_N^{(l)}$, and hence, via (2), on $\mathcal{G}_N^{(l)}$. The associated graded for the weight and depth filtrations $\text{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$ is a Lie algebra bigraded by negative integers $-w$ and $-m$. We call it the *level N Galois Lie algebra*. When $N = 1$ it is denoted by $\text{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}$.

The weight filtration on $\mathbb{L}_N^{(l)}$ admits a splitting, i.e. is defined by a grading, compatible with the depth filtration. Such a weight splitting is provided by the eigenspaces of a Frobenius $F_p \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, ($p \nmid N$). Therefore taking Gr for the weight and depth filtrations we get an embedding

$$\text{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \hookrightarrow \text{Gr} \text{Der} \mathbb{L}_N^{(l)}$$

The vector space $\text{Gr} \mathcal{G}_{-w, -m}^{(l)}(\mu_N)$ is nonzero only if $w \geq m \geq 1$. As a $\text{Gal}(\mathbb{Q}(\zeta_{l^\infty})/\mathbb{Q})$ -module it is isomorphic to a direct sum of copies of $\mathbb{Q}_l(w)$.

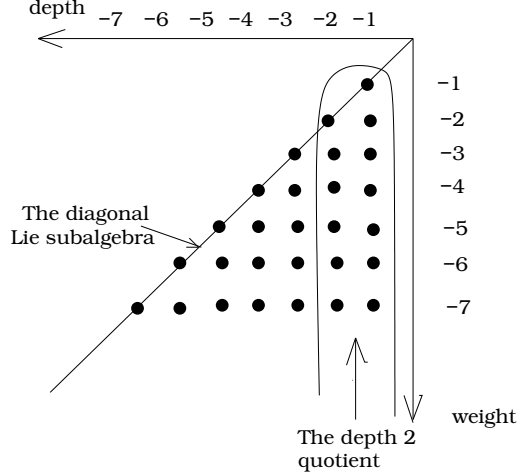
For every integer $m \geq 1$ we have the depth m quotient

$$\text{Gr} \mathcal{G}_{\bullet, \geq -m}^{(l)}(\mu_N) := \frac{\text{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)}{\text{Gr} \mathcal{G}_{\bullet, < -m}^{(l)}(\mu_N)}$$

It is a bigraded pro-nilpotent Lie algebra of the nilpotence class $\leq m$. In particular $\text{Gr} \mathcal{G}_{\bullet, \geq -1}^{(l)}(\mu_N)$ is abelian.

The direct sum of the components of the Galois Lie algebra satisfying weight = depth condition is a Lie subalgebra $\text{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$ called the *diagonal Galois Lie algebra*.

The shape of the Galois Lie algebra:



Recall the standard cochain complex of a Lie algebra \mathcal{G}

$$\mathcal{G}^\vee \xrightarrow{\delta} \Lambda^2 \mathcal{G}^\vee \xrightarrow{\delta} \Lambda^3 \mathcal{G}^\vee \longrightarrow \dots$$

where the first differential is dual to the commutator map $[\cdot, \cdot] : \Lambda^2 \mathcal{G} \longrightarrow \mathcal{G}$, and the others are obtained using the Leibniz rule. The condition $\delta^2 = 0$ is equivalent to the Jacobi identity. If the Lie algebra \mathcal{G} is graded its standard cochain complex inherits the grading.

Let V_m be the standard m -dimensional representation of GL_m . Our main goal is to show that

the depth m , weight w part of the standard cochain complex of the level N Galois Lie algebra is related to the geometry of the local system with the fiber $S^{w-m}V_m$ over the level N modular variety for the group GL_m/\mathbb{Q} .

In particular the structure of the depth m quotient of the Galois Lie algebra is described by geometry of the modular variety for GL_m/\mathbb{Q} .

For $m = 1$ this is deduced from the motivic theory of classical polylogarithms developed by Deligne and Beilinson ([D], [Be], [BD], [HW]). If $m = 1$ we deal with the scheme S_N , emphasizing that it is a modular variety for GL_1/\mathbb{Q} .

In this paper we prove general results about the Galois action on $\pi_1^{(l)}(X_N, v_\infty)$, and combining them with the results of [G2-3] establish the relationship above for $m = 2$ and essentially for $m = 3$. The results [G5,7] indicate a similar story for $m = 4$.

For example in the case $N = 1$ our main results describe the structure of the Lie algebra $\mathrm{Gr}\mathcal{G}_{\bullet, \geq -m}^{(l)}$ very explicitly for $m = 2, 3$: for $m = 2$ it is given

in terms of the classical modular triangulation of the hyperbolic plane (see the picture in s. 2.3) and for $m = 3$ via a similar $GL_3(\mathbb{Z})$ -equivariant structure of the symmetric space $\mathbb{H}_3 := GL_3(\mathbb{R})/O(3) \cdot \mathbb{R}_+^*$ defined in [G3] using Voronoi's decomposition of \mathbb{H}_3 . A detailed exposition of these results see in section 2.

We expect a similar description for all m in terms of the rank m modular complexes, see conjecture 2.9 below. For a reformulation without modular complexes see conjecture 1.2. Theorem 1.2 in [G3] shows that they are equivalent.

Let me say few words about the methods. The Lie algebra $\mathcal{G}_N^{(l)}$ enjoys the following properties:

- i) $\mathcal{G}_N^{(l)}$ acts by derivations of the Lie algebra $\mathbb{L}_N^{(l)}$.
- ii) There is a subspace $\mathbb{Q}_l(1) \subset \mathbb{L}_N^{(l)}$ trivial as a Galois module (“canonical loop around infinity”).
- iii) The Galois action preserves conjugacy classes of loops around 0 and μ_N .
- iv) The group μ_N acts on $\mathbb{L}_N^{(l)}$ commuting with the Galois action, see s. 3.1.
- v) The Galois action is compatible with the maps $X_{NM} \longrightarrow X_M$ given by $x \longmapsto x$ and $x \longmapsto x^N$, see s. 5.7.

We construct explicitly in s. 5 the Lie subalgebra of $\text{GrDer}\mathbb{L}_N^{(l)}$ respecting all these properties. One of our key points is that the Lie algebra $\text{Gr}\mathcal{G}_N^{(l)}$ should lie in a smaller Lie subalgebra which we single out by imposing

(!) The “power shuffle” relations.

To explain the origin of these relations let me recall the multiple polylogarithm functions [G9, 3]:

$$Li_{n_1, \dots, n_m}(x_1, \dots, x_m) := \sum_{0 < k_1 < \dots < k_m} \frac{x_1^{k_1} \dots x_m^{k_m}}{k_1^{n_1} \dots k_m^{n_m}}$$

Their immediate property is the shuffle product formula, which in the simplest case reads as follows:

$$Li_n(x)Li_m(y) = Li_{n,m}(x, y) + Li_{n+m}(xy) + Li_{m,n}(y, x)$$

Indeed,

$$\sum_{k_1, k_2 > 0} \frac{x_1^{k_1} x_2^{k_2}}{k_1^{n_1} k_2^{n_2}} = \left(\sum_{0 < k_1 < k_2} + \sum_{k_1 = k_2 > 0} + \sum_{0 < k_2 < k_1} \right) \frac{x_1^{k_1} x_2^{k_2}}{k_1^{n_1} k_2^{n_2}}$$

Similar arguments obviously lead to the general formula. Our most nontrivial constraint (!) on the image of the Galois group is an l -adic/motivic version of these relations, considered modulo the lower depth terms, which allows to avoid complications related to regularization of the divergent relations. Although the proof of shuffle product relations for multiple polylogarithms is so simple, it is rather difficult to deal with their l -adic/motivic version. We proved in s. 5 that imposing relations (!) and i)-v) we get a Lie subalgebra in $\text{GrDer}\mathbb{L}_N^{(l)}$. Its

precise description goes as follows. We construct in s. 5 (see theorem 5.2c)) an embedding

$$\xi_{\mu_N} : D_{\bullet\bullet}(\mu_N) \hookrightarrow \text{GrDer}\mathbb{L}_N$$

The Lie subalgebra $\xi_{\mu_N}(D_{\bullet\bullet}(\mu_N))$ coincides with the Lie algebra described by the six properties above. If we assume relations imposed by ii)-iv) there is an amazing duality between the relations (!) and i), which is naturally build in the definition of the dihedral Lie algebras, see Remark in s. 4.3.

Conjecture 1.1 $\text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \subset \xi_{\mu_N}(D_{\bullet\bullet}(\mu_N)) \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

If G is a trivial group we set $D_{\bullet\bullet} := D_{\bullet\bullet}(\{e\})$, and $\xi := \xi_{\{e\}}$.

Conjecture 1.2 *One has $\xi(D_{\bullet\bullet}) \otimes \mathbb{Q}_l = \text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}$.*

Since the Lie algebra $D_{\bullet\bullet}$ is defined *very* explicitly, this would give a precise description of the associated graded of the image of the Galois group.

The $(-w, -m)$ -component of $D_{\bullet\bullet}(G)$ can be nonzero only if $w \geq m \geq 1$. The *diagonal* Lie algebra $D_{\bullet\bullet}^{\Delta}(G)$ is the subalgebra of $D_{\bullet\bullet}(G)$ formed by the “depth = weight” components. It is graded by the weight.

Conjecture 1.3 *Let p be a prime number. Then $\text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_p) = \xi_{\mu_p}(D_{\bullet\bullet}^{\Delta}(\mu_p))$.*

In this paper we prove conjecture 1.1 for the depth 2 quotient in s. 7.3, and for the depth 3 quotient in the $N = 1$ case in s. 7.4. Its complete proof will appear elsewhere. We prove conjectures 1.2 and 1.3 in the depth 2 and 3. The relationship with geometry of modular varieties comes out of blue via the results of [G2-3, 5], and essentially used in the proofs. The results of [G5,7] indicate a possibility to extend the proofs for the depth 4.

We conjecture that the level N Galois Lie algebra is “very close” to the image of the dihedral Lie algebra of μ_N . For example the asymptotics of dimensions of their (w, m) -components when $N \rightarrow \infty$ should coincide.

For $N = 1$ there is another Lie subalgebra, the Grothendieck-Teichmuller Lie algebra \mathcal{Grt} defined by Drinfeld [Dr], sitting inside of $\text{Der}\mathbb{L}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, v_{\infty})$.

Conjecture 1.4 *The Lie subalgebra $\xi(D_{\bullet\bullet})$ coincides with the associated graded for the depth filtration of the Grothendieck-Teichmuller Lie algebra \mathcal{Grt}_1 .*

We do not even know that one of these Lie algebras contains the other.

We summarize the relationship between these Lie algebra in the following diagram, where the top left arrow assumes conjecture 1.1, and all the other arrows are embeddings.

$$\begin{array}{ccc} \xi(D_{\bullet\bullet}) \otimes \mathbb{Q}_l & & \\ \cup & \searrow & \\ \mathcal{G}^{(l)} & \hookrightarrow & \text{Gr}^D \text{Der}\mathbb{L}^{(l)}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, v_{\infty}) \\ \cap & \nearrow & \\ \text{Gr}^D \mathcal{Grt}_1 \otimes \mathbb{Q}_l & & \end{array}$$

In the next section, which continues the introduction, we formulate our results in detail.

2 Formulations of the main results

1. The Lie algebras of Galois symmetries of $\pi^{(l)}(X_N)$. For any group H , denote by $H(m)$ its lower central series for H :

$$H(1) := H, \quad H(m+1) := [H(m), H]$$

Here $[\cdot, \cdot]$ stands for the closure of the commutator subgroup.

The group $\pi_1^{(l)}(X_N) := \pi_1^{(l)}(X_N, v_\infty)$ has two filtrations by normal subgroups, indexed by integers $n \leq 0$:

i) *The weight filtration \mathcal{F}_\bullet^W* by the lower central series:

$$\mathcal{F}_{-w}^W \pi_1^{(l)}(X_N) := \pi_1^{(l)}(X_N)(w)$$

Let I_N be the kernel of the map $\pi_1^{(l)}(X_N) \longrightarrow \pi_1^{(l)}(\mathbb{G}_m)$ induced by the embedding $X_N \hookrightarrow \mathbb{G}_m$.

ii) *The depth filtration \mathcal{F}_\bullet^D* is given by the lower central series $I_N(m)$ for I_N :

$$\mathcal{F}_0^D \pi_1^{(l)}(X_N) := \pi_1^{(l)}(X_N), \quad \mathcal{F}_{-m}^D \pi_1^{(l)}(X_N) := I_N(m)$$

Notice that I_N is not profinitely generated, however since we will always divide by \mathcal{F}^W , this causes no problem.

These filtrations provide a projective system of quotients

$$\pi_1^{(l)}(X_N)_{[w,m]} := \pi_1^{(l)}(X_N) / \mathcal{F}_{-w-1}^W \cdot \mathcal{F}_{-m-1}^D \quad (3)$$

of the group $\pi_1^{(l)}(X_N)$. They are unipotent l -adic Lie groups. The Galois group acts by automorphisms of (3), so we are picking up homomorphisms

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}\left(\pi_1^{(l)}(X_N)_{[w+1,m+1]}\right) \quad (4)$$

The images of maps (4) sit in the middle of short exact sequences

$$0 \longrightarrow U_{N;[w,m]}^{(l)} \longrightarrow G_{N;[w,m]}^{(l)} \longrightarrow \mathrm{Gal}\left(\mathbb{Q}(\zeta_{l^\infty N})/\mathbb{Q}\right) \longrightarrow 0$$

In particular the map (4) for $m = w = 1$ is provided by the action of the Galois group on $H_{et}^1(X_N \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_l)(1)$. So its image is $\mathrm{Gal}(\mathbb{Q}(\zeta_{l^\infty N})/\mathbb{Q})$, and $U_{N;[0,0]}^{(l)}$ is the trivial group.

The weight and depth filtrations induce the two filtrations on the automorphism group of $\mathbb{L}_N^{(l)}$. Taking the quotients of the image of the Galois group

with respect to these filtrations we come to the groups $U_{N;[w,m]}^{(l)}$. The associated graded for the weight filtration is isomorphic to the free Lie algebra generated by $H_{et}^1(X_N \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_l)(1)$. Since $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ acts trivially on it $U_{N;[w,m]}^{(l)}$ is a unipotent l -adic Lie group of the nilpotence class $\leq m$.

There are natural projections

$$U_{N;[w,m]}^{(l)} \longrightarrow U_{N;[w',m']}^{(l)} \quad \text{if } w \geq w', m \geq m'$$

These groups form a projective system. Denote by $\text{Lie}(\ast)$ the corresponding Lie algebras over \mathbb{Q}_l , and consider the pronilpotent Lie algebra

$$\mathcal{G}_N^{(l)} := \varprojlim U_{N;[w,m]}^{(l)}$$

It is, by construction, bifiltered by the *weight* w and *depth* m . The associated graded $\text{Gr}_{\bullet\bullet}^{(l)}(\mu_N)$ is a Lie algebra bigraded by negative integers $-w$ and $-m$.

2. The depth 1 case as part of the general picture. Recall that the depth 1 quotient of the Galois Lie algebra is abelian. Therefore the result below, which follows from the motivic theory of classical polylogarithms developed by Deligne and Beilinson ([D], [BD], [HW]), settles the $m = 1$ case of our story.

Theorem 2.1 *There is canonical isomorphism*

$$\text{Gr}_{-w,-1}^{(l)}(\mu_N) = \text{Hom}\left(K_{2w-1}(S_N), \mathbb{Q}_l\right)$$

The right hand side is calculated by Borel's theorem [B]. The result fits in our general framework as follows. Then

$$K_{2w-1}(S_N) \otimes \mathbb{Q} = H^0(S_N(\mathbb{C}), \mathcal{L}_{S^{w-1}V_1})^+ \quad w > 1 \quad (5)$$

where $\mathcal{L}_{S^{w-1}V_1}$ is the local system on $S_N(\mathbb{C})$ corresponding to the GL_1 -module $S^{w-1}V_1$ and $+$ means invariants of the involution acting on $S_N(\mathbb{C})$ as the complex conjugation c , and on the local system via $v \mapsto -v$ on V_1 .

In particular for $N = 1$ we get

$$\dim \text{Gr}_{-w,-1}^{(l)} = \dim K_{2w-1}(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 0 & w \text{ even, or } w = 1 \\ 1 & w > 1 \text{ odd} \end{cases} \quad (6)$$

This has been known thanks to Soulé [So], Deligne [D] and Ihara [Ih2].

If we treat the pair $\{S_n(\mathbb{C}), c\}$ as a stack and $\mathcal{L}_{S^{w-1}V_1}$ as a local system on this stack then $(5) = H^0(\{S_n(\mathbb{C}), c\}, \mathcal{L}_{S^{w-1}V_1})$.

Let $\mathbb{A}_{\mathbb{Q}}$ be the adels of \mathbb{Q} . In general the depth m , weight w part of the standard cochain complex of the Lie algebra $\text{Gr}_{\bullet\bullet}^{(l)}(\mu_N)$ is related to the modular stack

$$Y_1(m; N) := GL_m(\mathbb{Q}) \backslash GL_m(\mathbb{A}_{\mathbb{Q}}) / K_1(m; N) \cdot \mathbb{R}_+^* \cdot O_m \quad (7)$$

It is given by the quotient of

$$GL_m(\mathbb{Q}) \backslash GL_m(\mathbb{A}_{\mathbb{Q}}) / K_1(m; N) \cdot \mathbb{R}_+^* \cdot SO_m \quad (8)$$

by the group $\mathbb{Z}/2\mathbb{Z} = O(m)/SO(m)$ acting on the right. Here the subgroup $K_1(m; N) \subset \prod_p GL_m(\mathbb{Z}_p)$ is defined by imposing congruence conditions at the primes $p|N$. Namely, if $N = \prod p^{v_p(N)}$ then its p -component consists of the elements of $GL_m(\mathbb{Z}_p)$ whose last row is congruent to $(0, \dots, 0, 1) \bmod p^{v_p(N)}$.

Example. When $m = 1$ the set (8) is isomorphic to $S_N(\mathbb{C})$ and $O(1)$ acts as the complex involution c . For $m > 1$ the stack (7) coincides with the modular variety (1). When $m = 2$ one has $GL_2(\mathbb{R})/R_+^* \cdot SO(2) = \mathbb{C} - \mathbb{R}$ and $O(2)/SO(2)$ acts as the complex conjugation $z \mapsto \bar{z}$.

Let us go beyond the depth 1 case. We start from the $N = 1$ case. The following result generalizes the first line of (6):

Theorem 2.2 $\dim \mathcal{G}_{-w, -m}^{(l)} = 0$ if $w + m$ is odd.

3. The Galois action on $\pi_1^{(l)}(\mathbb{P}^1 - \{0, 1, \infty\}, v_\infty)$: a description of the depth 2 quotient via the modular triangulation of the hyperbolic plane. The structure of the Lie algebra $\mathcal{G}_{\bullet, \geq -2}^{(l)}$ is completely described by the Lie commutator map

$$[,] : \Lambda^2 \mathcal{G}_{\bullet, -1}^{(l)} \longrightarrow \mathcal{G}_{\bullet, -2}^{(l)} \quad (9)$$

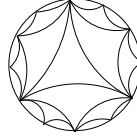
The left hand side is known to us by (6). So to describe (9) one needs to define the right hand side and the commutator map. Dualizing (9) we get the depth two piece of the standard cochain complex of the Lie algebra $\mathcal{G}_{\bullet, \geq -2}^{(l)}$:

$$\mathcal{G}_{\bullet, -2}^{(l)} \overset{\vee}{\longrightarrow} \Lambda^2 \mathcal{G}_{\bullet, -1}^{(l)} \overset{\vee}{\longrightarrow} \quad (10)$$

To describe it we need to introduce the following two complexes:

$$M_{(2)}^* := M_{(2)}^1 \longrightarrow M_{(2)}^2 \quad \text{and} \quad \mathbb{M}_{(2)}^* := M_{(2)}^1 \longrightarrow M_{(2)}^2 \longrightarrow M_{(2)}^3 \quad (11)$$

The first one is the chain complex of the classical modular triangulation



of the hyperbolic plane \mathbb{H}_2 where the central ideal triangle has vertices at $0, 1$ and ∞ . We place it in degrees $[1, 2]$. For example $M_{(2)}^1$ is the group generated by the triangles. The second complex is the chain complex of the modular triangulation of the hyperbolic plane *extended by cusps*, i.e. by $\mathbb{P}^1(\mathbb{Q})$. We place it in degrees $[1, 3]$.

The group $GL_2(\mathbb{R})$, acting on $\mathbb{C} - \mathbb{R}$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$, commutes with $z \mapsto \bar{z}$. We let $GL_2(\mathbb{R})$ act on the upper half plane \mathbb{H}_2 by identifying \mathbb{H}_2 with the quotient of $\mathbb{C} - \mathbb{R}$ by complex conjugation. The action of the subgroup $GL_2(\mathbb{Z})$ preserves the modular picture. So $M_{(2)}^*$ and $\mathbb{M}_{(2)}^*$ are complexes of $GL_2(\mathbb{Z})$ -modules.

A digression on cohomology of a subgroup Γ of $GL_2(\mathbb{Z})$. Let V be a GL_2 -module. For a torsion free subgroup Γ of $GL_2(\mathbb{Z})$ the group cohomology $H^*(\Gamma, V)$ are isomorphic to the cohomology of $\Gamma \backslash \mathbb{H}_2$ with coefficients in the local system \mathcal{L}_V corresponding to V . Notice that $H^*(\Gamma \backslash \mathbb{H}_2, \mathcal{L}_V) = H^*(\Gamma \backslash \mathbb{H}_2, Rj_* \mathcal{L}_V)$ where $j : \Gamma \backslash \mathbb{H}_2 \hookrightarrow \overline{\Gamma \backslash \mathbb{H}_2}$.

For a torsion free finite index subgroup Γ of $GL_2(\mathbb{Z})$ one defines the cuspidal cohomology $H_{\text{cusp}}^*(\Gamma, V)$ as the cohomology of $\overline{\Gamma \backslash \mathbb{H}_2}$ with coefficients in a middle extension of the local system \mathcal{L}_V . In our case the middle extension means the sheaf $j_* \mathcal{L}_V$.

For any finite index subgroup $\Gamma \hookrightarrow GL_2(\mathbb{Z})$ there is a normal torsion free finite index subgroup $\tilde{\Gamma} \hookrightarrow \Gamma$. So if V is a \mathbb{Q} -rational GL_2 -module one has, using the Hochschild-Serre spectral sequence,

$$H^*(\Gamma, V) = H^*(\tilde{\Gamma}, V)^{\Gamma/\tilde{\Gamma}} \quad (12)$$

We define the cuspidal cohomology for arbitrary finite index subgroup Γ by reducing it to the torsion free case by formula similar to (12).

Lemma 2.3 *Let Γ be a finite index subgroup of $GL_2(\mathbb{Z})$ and V a \mathbb{Q} -rational GL_2 -module. Then the complex*

$$M_{(2)}^* \otimes_{\Gamma} V[1] \quad (13)$$

computes the cohomology $H^(\Gamma, V \otimes \varepsilon_2)$, and the complex*

$$\mathbb{M}_{(2)}^* \otimes_{\Gamma} V[1] \quad (14)$$

computes the cuspidal cohomology $H_{\text{cusp}}^(\Gamma, V \otimes \varepsilon_2)$.*

Proof. The complex (13), up to a shift, is the complex of chains with coefficients in the local system \mathcal{L}_V ; it is relative to the modular triangulation of $\overline{\Gamma \backslash \mathbb{H}_2}$, which has finitely many cells, with $\Gamma \backslash \mathbb{H}_2$ a union of open cells, hence giving homology groups for locally finite chains (the Borel-Moore homology).

If we take the dual for the complex (13), as well as for (14), we obtain a cochain complex computing the cohomology of $\overline{\Gamma \backslash X}$ with coefficients in a sheaf: the sheaf is $j_* \mathcal{L}_{V^\vee}$ for (13) and the middle extension $j_* \mathcal{L}_{V^\vee}$ for (14) (notice that invariants by the stabilizer of the cusp dual to the coinvariant group).

For the complex itself by Poincaré duality we get cohomology of $\Gamma \backslash \mathbb{H}_2$ (respectively $\overline{\Gamma \backslash \mathbb{H}_2}$) with value in the local system \mathcal{L}_V twisted by the orientation class, i.e. $\mathcal{L}_{V \otimes \varepsilon_2}$ (respectively its middle extension). The lemma is proved.

To clarify the homological algebra meaning of the modular complex consider complex $\mathcal{M}_{-1}^{(2)} \xleftarrow{\partial} \mathcal{M}_{-2}^{(2)}$, where $\mathcal{M}_{-*}^{(2)} := M_{(2)}^*$ and ∂ is dual to the differential in the complex $M_{(2)}^*$. Then $M_{(2)}^*$ is a subcomplex of $\text{Hom}(\mathcal{M}_{-*}^{(2)}, \mathbb{Z})$. As a complex of $SL_2(\mathbb{Z})$ -modules $\mathcal{M}_{(2)}^*$ is the chain complex of the tree dual to the modular triangulation, shifted by 1. As a complex of $GL_2(\mathbb{Z})$ -modules it is a resolution of $\varepsilon_2[1]$. The stabilizers of the action of $GL_2(\mathbb{Z})$ on the sets of triangles and edges of the modular triangulation are finite. Thus for any \mathbb{Q} -rational GL_2 -module V and any subgroup $\Gamma \subset GL_2(\mathbb{Z})$

$$\text{Hom}_{\Gamma}(\mathcal{M}_{-*}^{(2)}, V) \quad \text{computes} \quad H^{*-1}(\Gamma, V)$$

Let us return to description of the image of the Galois group.

Let $\mathcal{G}_{\bullet\bullet}$ be any bigraded Lie algebra. Consider the bigraded Lie algebra

$$\widehat{\mathcal{G}}_{\bullet\bullet} := \mathcal{G}_{\bullet\bullet} \oplus \mathbb{Q}(-1, -1) \quad (15)$$

where $\mathbb{Q}(-1, -1)$ is a one dimensional Lie algebra of the bidegree $(-1, -1)$. The standard cochain complex of $\widehat{\mathcal{G}}_{\bullet\bullet}$ admits a canonical decomposition

$$\Lambda^* \widehat{\mathcal{G}}_{\bullet\bullet}^{\vee} = \Lambda^* \mathcal{G}_{\bullet\bullet}^{\vee} \oplus \Lambda^* \mathcal{G}_{\bullet\bullet}^{\vee} \otimes \mathbb{Q}(1, 1) \quad (16)$$

Strangely enough it is simpler to describe the structure of the Lie algebra $\widehat{\mathcal{G}}_{\bullet, \geq -2}^{(l)}$ than $\mathcal{G}_{\bullet, \geq -2}^{(l)}$. The Lie algebra structure of $\widehat{\mathcal{G}}_{\bullet, \geq -2}^{(l)}$ is completely described by the commutator map

$$[\cdot, \cdot] : \Lambda^2 \widehat{\mathcal{G}}_{\bullet, -1}^{(l)} \longrightarrow \mathcal{G}_{\bullet, -2}^{(l)} \quad (17)$$

Theorem 2.4 *a) The weight w part of the complex*

$$\mathcal{G}_{\bullet, -2}^{(l)\vee} \xrightarrow{\delta} \Lambda^2 \widehat{\mathcal{G}}_{\bullet, -1}^{(l)\vee} \quad (18)$$

dual to (17), is isomorphic to the complex

$$\left(M_{(2)}^* \otimes_{GL_2(\mathbb{Z})} S^{w-2} V_2 \right) \otimes \mathbb{Q}_l \quad (19)$$

b)

$$\dim \text{Gr} \mathcal{G}_{-w, -2}^{(l)} = \begin{cases} 0 & w : \text{ odd} \\ \left[\frac{w-2}{6} \right] & w : \text{ even} \end{cases} \quad (20)$$

Using the part a) and lemma 2.3 we compute the Euler characteristic of the complex (18). Then we use (6) to get formula (20). A Hodge-theoretic version of theorem 2.4 appeared first as theorem 7.2 in [G1]. The estimate $\dim \mathcal{G}_{-w, -2}^{(l)} \geq \left[\frac{w-2}{6} \right]$ has been independently obtained by Ihara and Takao ([Ih3]).

In particular there is a *canonical* isomorphism

$$m_1^{(l)} : \mathcal{G}_{-w,-2}^{(l)}{}^\vee \xrightarrow{=} \left(M_{(2)}^1 \otimes_{GL_2(\mathbb{Z})} S^{w-2} V_2 \right) \otimes \mathbb{Q}_l \quad (21)$$

providing a description of the vector space $\mathcal{G}_{-w,-2}^{(l)}$. To describe the isomorphism of complexes from theorem 2.4 we need to define the map

$$m_2^{(l)} : \Lambda_w^2 \left(\widehat{\mathcal{G}}_{\bullet,-1}^{(l)} \right)^\vee \xrightarrow{=} \left(M_{(2)}^2 \otimes_{GL_2(\mathbb{Z})} S^{w-2} V_2 \right) \otimes \mathbb{Q}_l \quad (22)$$

where Λ_w^2 the weight w part of Λ^2 . We do it as follows. The stabilizer in $GL_2(\mathbb{Z})$ of the geodesic from 0 to $i\infty$ on the upper half plane is generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Therefore the right hand side of (22) is identified with the space of degree $w-2$ polynomials $f(t_1, t_2)$, skewsymmetric in the variables t_1, t_2 and of even degree in each of them: $f(t_1, t_2) = -f(t_2, -t_1) = f(-t_1, t_2)$.

Let $\zeta_{\mathcal{M}}^{(l)}(n)$ for $n > 1$ be Soulé's generator of $\left(\text{Gr} \mathcal{G}_{-2n+1,-1}^{(l)} \right)^\vee$, see section 6.

We define $\zeta_{\mathcal{M}}^{(l)}(1)$ as a generator of $\mathbb{Q}_l(-1, -1) = \left(\text{Gr} \mathcal{G}_{-1,-1}^{(l)} \right)^\vee$ and set

$$m_2^{(l)} : \zeta_{\mathcal{M}}^{(l)}(2m+1) \wedge \zeta_{\mathcal{M}}^{(l)}(2n+1) \longmapsto t_1^{2m} t_2^{2n} - t_1^{2n} t_2^{2m}$$

The map $m_2^{(l)}$ identifies $\Lambda^2 \left(\mathcal{G}_{\bullet,-1}^{(l)} \right)^\vee$ with the subspace of polynomials $f(t_1, t_2)$, skewsymmetric in t_1, t_2 and of *positive* even degree in each of them. Thus we get a precise description of the Lie algebra $\mathcal{G}_{\bullet, \geq -2}^{(l)}$ as well.

The decomposition (16) of complex (18) corresponds to decomposition of complex (19) into a direct sum of the subcomplex computing the (truncated) cuspidal cohomology, and the subcomplex, consisting of a single group in the degree 2, computing the Eisenstein part of the cohomology.

More precisely, consider the truncated complex

$$\begin{aligned} \tau_{[1,2]} \left(\mathbb{M}_{(2)}^* \otimes_{GL_2(\mathbb{Z})} V \right) &:= \\ M_{(2)}^1 \otimes_{GL_2(\mathbb{Z})} V &\longrightarrow \text{Ker} \left(M_{(2)}^2 \otimes_{GL_2(\mathbb{Z})} V \longrightarrow M_{(2)}^3 \otimes_{GL_2(\mathbb{Z})} V \right) \end{aligned} \quad (23)$$

It is a subcomplex of the complex (13).

Theorem 2.5 *The complex*

$$\tau_{[1,2]} \left(\mathbb{M}_{(2)}^* \otimes_{GL_2(\mathbb{Z})} S^{w-2} V_2 \right) \otimes \mathbb{Q}_l \quad (24)$$

is canonically isomorphic to the weight w part of the complex (10).

4. The Galois action on $\pi_1^{(l)}(\mathbb{P}^1 - \{0, 1, \infty\}, v_\infty)$: the depth 3 quotient.

To describe the structure of the Lie algebra $\mathcal{G}_{\bullet, \geq -3}^{(l)}$ we need only to define the commutator map

$$[,] : \mathcal{G}_{\bullet, -2}^{(l)} \otimes \mathcal{G}_{\bullet, -1}^{(l)} \longrightarrow \mathcal{G}_{\bullet, -3}^{(l)} \quad (25)$$

obeying the Jacoby identity. Indeed, the commutator $[,] : \Lambda^2 \mathcal{G}_{\bullet, -1}^{(l)} \longrightarrow \mathcal{G}_{\bullet, -2}^{(l)}$ has been described in the part a) of theorem 2.4. Dualizing one sees that we need to describe the depth three part of the standard cochain complex of $\mathcal{G}_{\bullet, \geq -3}^{(l)}$:

$$\mathcal{G}_{\bullet, -3}^{(l) \vee} \longrightarrow \mathcal{G}_{\bullet, -2}^{(l) \vee} \otimes \mathcal{G}_{\bullet, -1}^{(l) \vee} \longrightarrow \Lambda^3 \mathcal{G}_{\bullet, -1}^{(l) \vee} \quad (26)$$

The first map is dual to the map (25). Complex (26) is described in s. 1.7. Here is an important corollary:

Theorem 2.6 a) *The complex (26) computes the cuspidal cohomology*

$$H_{\text{cusp}}^i(GL_3(\mathbb{Z}), S^{w-3}V_3) \quad \text{at } i = 1, 2, 3$$

b) *The complex (26) is acyclic.*

c)

$$\dim \text{Gr} \mathcal{G}_{-w, -3}^{(l)} = \begin{cases} 0 & w : \text{ even} \\ \left[\frac{(w-3)^2 - 1}{48} \right] & w : \text{ odd} \end{cases} \quad (27)$$

By b) the Euler characteristic of the complex (26) is zero. So using (6) and theorem 2.4b) we get formula (27).

Remark. Compare theorems 2.4, 2.6 with theorems 1.4, 1.5 in [G3] where the dimension of the \mathbb{Q} -space of reduced multiple ζ -values of depth 2 and 3 is estimated from *above* by (20) and (27). According to some standard conjectures in arithmetic algebraic geometry one should have

$$\mathcal{G}_{-w, -m}^{(l)} = (\text{the space of reduced multiple } \zeta\text{'s of weight } -w, \text{ depth } -m) \otimes_{\mathbb{Q}} \mathbb{Q}_l$$

This supplies an additional evidence for the statement that the estimates given in theorems 1.4, 1.5 in [G3] are exact. Notice that computation of the dimension of the \mathbb{Q} -space of reduced multiple ζ -values seems to be a transcendently difficult problem (we can not prove that $\zeta(5) \notin \mathbb{Q}!$), while its more sophisticated l -adic analog is easier to approach.

To describe results and conjectures about the structure of the Lie algebra $\text{Gr} \mathcal{G}_{\bullet, \bullet}^{(l)}$ we need to recall the definition of the modular complexes given in the section 5 of [G3].

5'. The modular complexes. Let L_m be a rank m lattice. The modular complex $M_{(m)}^*$ is a complex of left $\text{Aut}(L_m) = GL_m(\mathbb{Z})$ -modules which sits in the degrees $[1, m]$ and looks as follows:

$$M_{(m)}^1 \xrightarrow{\partial} M_{(m)}^2 \xrightarrow{\partial} \dots \xrightarrow{\partial} M_{(m)}^m$$

By definition $M_{(1)}^1$ is the trivial $GL_1(\mathbb{Z})$ -module. When $m = 2$ it is isomorphic to the complex on the left in (11). In general its definition is purely combinatorial. We recall it now.

i) *Extended basis and homogeneous affine basis of a lattice.* We say that an *extended basis* of a lattice L_m is an $(m+1)$ -tuple of vectors v_1, \dots, v_{m+1} of the lattice such that $v_1 + \dots + v_{m+1} = 0$ and v_1, \dots, v_m is a basis. Then omitting any of the vectors v_1, \dots, v_{m+1} we get a basis of the lattice.

The group $GL_m(\mathbb{Z})$ acts from the left on the set of basis of L_m , considered as columns of vectors (v_1, \dots, v_m) . This provides the set of extended basis with the structure of the left principal homogeneous space for $GL_m(\mathbb{Z})$.

Let u_1, \dots, u_{m+1} be elements of the lattice L_m such that the set of elements $\{(u_i, 1)\}$ form a basis of $L_m \oplus \mathbb{Z}$. The lattice L_m acts on such sets by $l : \{(u_i, 1)\} \mapsto \{(u_i + l, 1)\}$. We call the coinvariants of this action *homogeneous affine basis* of L_m and denote them by $\{u_1 : \dots : u_{m+1}\}$. Notice that $\{u_1 : \dots : u_{m+1}\}$ is a homogeneous affine basis if and only if $\{u_2 - u_1, u_3 - u_2, \dots, u_1 - u_{m+1}\}$ is an extended basis. So there is a canonical bijection

$$\text{homogeneous affine basis of } L_m \quad < - > \quad \text{extended basis of } L_m \quad (28)$$

$$\{u_1 : \dots : u_{m+1}\} \quad < - > \quad \{u_2 - u_1, u_3 - u_2, \dots, u_1 - u_{m+1}\}$$

ii) *The group $M_{(m)}^1$.* The abelian group $M_{(m)}^1$ is generated by the elements $< v_1, \dots, v_{m+1} >$ corresponding to extended basis v_1, \dots, v_{m+1} of L_m . To list the relations we need another set of the generators corresponding to the homogeneous affine basis of L_m via (28):

$$< u_1 : \dots : u_{m+1} > := < u'_1, u'_2, \dots, u'_{m+1} >, \quad u'_i := u_{i+1} - u_i$$

Let $\Sigma_{p,q}$ is the set of all shuffles of the ordered sets $\{1, \dots, p\}$ and $\{p+1, \dots, p+q\}$.

Relations. For $m = 1$ we have $< v_1, v_2 > = < v_2, v_1 >$.

For any $1 \leq k \leq m$ one has:

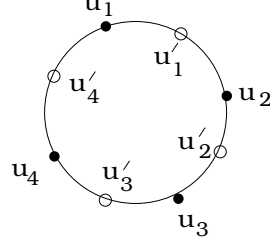
$$\sum_{\sigma \in \Sigma_{k, m-k}} < v_{\sigma(1)}, \dots, v_{\sigma(m)}, v_{m+1} > = 0 \quad (29)$$

$$\sum_{\sigma \in \Sigma_{k, m-k}} < u_{\sigma(1)} : \dots : u_{\sigma(m)} : u_{m+1} > = 0 \quad (30)$$

Theorem 2.7 *The double shuffle relations (29), (30) imply the following dihedral symmetry relations for $m \geq 2$:*

$$\begin{aligned} < v_1, \dots, v_m, v_{m+1} > &= < v_2, \dots, v_{m+1}, v_1 > = \\ < -v_1, \dots, -v_m, -v_{m+1} > &= (-1)^{m+1} < v_{m+1}, v_m, \dots, v_1 > \end{aligned}$$

We picture both types of the generators on the circle as shown on the picture:



Vectors of homogeneous affine basis are outside, and vectors of extended basis are inside of the circle (compare with s. 4.1 below).

iii) *The group $M_{(m)}^k$.* For each decomposition of L_m as a direct sum of k non zero lattices L^i , we consider the tensor product of the $M^1(L^i)$. We consider $M^1(L^i)$ as odd, and use the sign rule to identify $\otimes M^1(L^i)$ and $\otimes M^1(L^{i'})$ when the decomposition L and L' differ only in the ordering of the factors. The group $M_{(m)}^k$ is defined to be the sum over all such unordered decomposition $L_m = \oplus L^i$ of the corresponding $\otimes M^1(L^i)$. In other words it is generated by the elements $\langle A_1 \rangle \wedge \dots \wedge \langle A_k \rangle$ where A_i is an extended basis of the sublattice L_i and $\langle A_i \rangle$'s anticommute. Let us set

$$[v_1, \dots, v_k] := \langle v_1, \dots, v_k, v_{k+1} \rangle, \quad v_1 + \dots + v_k + v_{k+1} = 0$$

We define a homomorphism $\partial : M_{(m)}^1 \longrightarrow M_{(m)}^2$ by setting $\partial = 0$ if $m = 1$ and

$$\partial : \langle v_1, \dots, v_{m+1} \rangle \longmapsto -\text{Cycle}_{m+1} \left(\sum_{k=1}^{m-1} [v_1, \dots, v_k] \wedge [v_{k+1}, \dots, v_m] \right)$$

where the indices are modulo $m+1$. We get the differential in $M_{(m)}^\bullet$ by extending ∂ using the Leibniz rule:

$$\partial([A_1] \wedge [A_2] \wedge \dots) := \partial([A_1]) \wedge [A_2] \wedge \dots - [A_1] \wedge \partial([A_2]) \wedge \dots + \dots$$

Theorem 2.8 *The map ∂ is well defined and $\partial^2 = 0$, so we get a complex.*

Remark. Both the definition of the modular complex and these proofs are very similar to the definitions and basic properties of the dihedral Lie algebras discussed in detail in the section 4 below. Yet I do not know a general framework unifying them.

Remark. Let A be an arbitrary commutative ring. Then replacing L_m by a free rank m A -module one can construct modular complexes corresponding to the ring A , recovering the construction above when $A = \mathbb{Z}$.

6. A description of the Lie algebra $\text{Gr}_{\bullet\bullet}^{\widehat{\mathcal{G}}^{(l)}}$ via the modular complex. Set $V_m := L_m \otimes \mathbb{Q}$.

Conjecture 2.9 *There exists a canonical isomorphism between the complex*

$$\left(M_{(m)}^* \otimes_{GL_m(\mathbb{Z})} S^{w-m} V_m \right) \otimes_{\mathbb{Q}} \mathbb{Q}_l \quad (31)$$

and the depth m , weight w part of the standard cochain complex of the Lie algebra $\text{Gr}\widehat{\mathcal{G}}_{\bullet\bullet}^{(l)}$.

Example. The rank 1 modular complex $M_{(1)}^1$ is the trivial $GL_1(\mathbb{Z})$ -module \mathbb{Z} . Thus

$$M_{(1)}^1 \otimes_{GL_1(\mathbb{Z})} S^{w-1} V_1 = \begin{cases} 0 & w \text{ even} \\ \mathbb{Q} & w \text{ odd} \end{cases}$$

Thus formula (31) for $m = 1$ is just equivalent to (6) plus $\dim \text{Gr}\widehat{\mathcal{G}}_{-1,-1}^{(l)} = 1$. For $m = 2$ we get the description of the depth two part given in theorem 2.4.

Theorem 2.10 *There exists a canonical isomorphism between the complex*

$$\left(M_{(3)}^* \otimes_{GL_3(\mathbb{Z})} S^{w-3} V_3 \right) \otimes_{\mathbb{Q}} \mathbb{Q}_l \quad (32)$$

and the depth 3, weight w part of the standard cochain complex of the Lie algebra $\widehat{\mathcal{G}}_{\bullet,\geq-3}^{(l)}$.

Theorem 2.11 [G3] *There rank 3 modular complex is quasiisomorphic to the chain complex of the Voronoi decomposition of the symmetric space \mathbb{H}_3 for $GL_3(\mathbb{R})$, truncated in the degrees $[1, 3]$.*

This is theorem 6.2 in [G3]. To prove it we constructed a geometric realization of the rank 3 modular complex as a subcomplex of the truncated Voronoi complex.

On the geometric realization of the rank m modular complex see in [G5-7]. Using it we proved that the rank 4 modular complex, shifted by $[-2]$, is quasiisomorphic to chain complex of the Voronoi decomposition of the symmetric space \mathbb{H}_4 , truncated in the degrees $[3, 6]$.

7. The diagonal Galois Lie algebras and modular complexes. The diagonal Lie algebra $\text{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N)$ is a Lie subalgebra of the Galois Lie algebra $\text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$:

$$\text{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N) := \bigoplus_{w \geq 1} \text{Gr}\mathcal{G}_{-w,-w}^{(l)}(\mu_N) \hookrightarrow \text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$$

It is graded by the weight.

Equivalently, one can define the diagonal Galois Lie algebra as a Lie subalgebra of $\text{Gr}^W \mathcal{G}^{(l)}(\mu_N)$ by imposing the depth $\leq -w$ condition on each weight $-w$ subquotient:

$$\text{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N) := \bigoplus_{w \geq 1} \mathcal{F}_{-w}^D \text{Gr}_{-w}^W \mathcal{G}^{(l)}(\mu_N) \hookrightarrow \text{Gr}^W \mathcal{G}^{(l)}(\mu_N)$$

Since the Lie algebra $\text{Gr}^W \mathcal{G}^{(l)}(\mu_N)$ is non canonically isomorphic to the Lie algebra $\mathcal{G}^{(l)}(\mu_N)$, the diagonal Galois Lie algebra is isomorphic, although non canonically, to a certain Lie subalgebra of $\mathcal{G}^{(l)}(\mu_N)$. This is an important difference between the Galois Lie algebra and its diagonal part: the Galois Lie algebra in general is not isomorphic to $\mathcal{G}^{(l)}(\mu_N)$.

Theorem 2.12 *There is canonical isomorphism*

$$\text{Gr}\mathcal{G}_{-1,-1}^{(l)}(\mu_N) = \text{Hom}_{\mathbb{Q}}\left(\text{the group of the cyclotomic units in } \mathbb{Z}[\zeta_N][\frac{1}{N}], \quad \mathbb{Q}_l\right)$$

This rather elementary result is a particular case of theorem 2.1 for $w = 1$. It suggested the name “higher cyclotomy” for our story. Set

$$E_m(N) := \mathbb{Z}[\Gamma_1(m; N) \backslash GL_m(\mathbb{Z})] \quad \text{if } m > 1 \quad E_1(N) := \mathbb{Z}/N\mathbb{Z}$$

Conjecture 2.13 *Let p be a prime number. Then there exists a canonical isomorphism between the complex*

$$\left(M_{(m)}^* \otimes_{GL_m(\mathbb{Z})} E_m(N)\right) \otimes \mathbb{Q}_l \quad (33)$$

and the depth m part of the standard cochain complex of $\text{Gr}\widehat{\mathcal{G}}_{\bullet}^{(l)}(\mu_p)$.

Notice that if $m > 1$ then

$$M_{(m)}^* \otimes_{\Gamma_1(m; N)} \mathbb{Q} = M_{(m)}^* \otimes_{GL_m(\mathbb{Z})} E_m(N)$$

The depth ≥ -2 quotient of $\text{Gr}_{\bullet}^{(l)}\widehat{\mathcal{G}}(\mu_p)$ is described by the commutator map

$$[\cdot, \cdot] : \quad \Lambda^2 \text{Gr}\widehat{\mathcal{G}}_{-1}^{(l)}(\mu_p) \longrightarrow \text{Gr}\mathcal{G}_{-2}^{(l)}(\mu_p) \quad (34)$$

Let $\Gamma_1(p) := \Gamma_1(2; p) \cap SL_2(\mathbb{Z})$. Consider the modular curve $Y_1(p) := \Gamma_1(p) \backslash \mathbb{H}_2$. Projecting the modular triangulation of the hyperbolic plane onto $Y_1(p)$ we get the modular triangulation of $Y_1(p)$. The complex involution acts on the modular curve preserving the triangulation. Consider the following complex

$$\left(\text{the chain complex of the modular triangulation of } Y_1(p)\right)^+ \quad (35)$$

Here $+$ means the invariants of the action of the complex involution.

Theorem 2.14 *The dual to complex (34) is naturally **isomorphic** to complex (35).*

This is a depth two analog of theorem 2.12. In particular there is canonical isomorphism

$$\mathbb{Q}_l[\text{triangles of the modular triangulation of } Y_1(p)]^+ = \left(\text{Gr}\mathcal{G}_{-2}^{(l)}(\mu_p) \right)^\vee \quad (36)$$

It turns out that the subcomplex

$$\tau_{[1,2]} \left(\mathbb{M}_{(2)}^* \otimes_{\Gamma_1(2;p)} \mathbb{Q}_l \right) \hookrightarrow M_{(2)}^* \otimes_{\Gamma_1(2;p)} \mathbb{Q}_l$$

corresponds to the maximal quotient of the Galois group acting on $\pi^{(l)}(X_p)$, of weight=depth ≥ -2 , which is *unramified at* $1 - \zeta_p$, see section 7.9.

Now we turn to the depth three case. Let $\text{Gr}\mathcal{L}_{\bullet}^{(l)}(\mu_N) \hookrightarrow \text{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N)$ be the Lie subalgebra generated by $\text{Gr}\mathcal{L}_{-1}^{(l)}(\mu_N)$.

Theorem 2.15 *a) Conjecture 2.13 is valid for $m = 3$ for the Lie algebra $\widehat{\text{Gr}\mathcal{L}_{\bullet}^{(l)}}(\mu_p)$. So there exists canonical isomorphism between the complex (33) and the depth 3 part of the standard cochain complex of this Lie algebra.*

b) Let us assume conjecture 1.1 below. Then $\text{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p) = \text{Gr}\mathcal{G}_{-3}^{(l)}(\mu_p)$, and therefore conjecture 2.13 would be valid for $m = 3$.

There is a description of the vector space $\text{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p)$ similar to (36) in terms of certain 4-cells on the 5-dimensional modular variety $\Gamma_1(3;p) \backslash \mathbb{H}_3$ which were constructed in [G3]. See [G5-7] for a construction of certain $2(m-1)$ -cells in \mathbb{H}_m which should play a similar role in general.

Corollary 2.16 *If $p \geq 5$ then*

$$\begin{aligned} \dim \text{Gr}\mathcal{G}_{-2}^{(l)}(\mu_p) &= \frac{(p-1)(p-5)}{12} \\ \dim \text{Gr}\mathcal{G}_{-3}^{(l)}(\mu_p) \geq \dim \text{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p) &= \frac{(p-5)(p^2-2p-11)}{48} \end{aligned}$$

9. The structure of the paper. In the section 3 we explain how to linearize the action of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ on $\pi_1^{(l)}(X_N, v_\infty)$. We show that the Lie algebra of the image of the Galois group acts by the so-called special equivariant derivations of $\mathbb{L}^{(l)}(X_N, v_\infty)$, preserving the two filtrations.

In the section 4 we define the dihedral Lie coalgebra $\mathcal{D}_{\bullet\bullet}(G)$ of a commutative group G and derive its main properties. Completely similar arguments settle the basic properties of the modular complexes.

In the section 5 we study the Lie algebra $\text{Der}^{SE}L(G)$ of special equivariant derivations of the free Lie algebra $L(G)$ generated by the set $\{0\} \cup G$. We realize the dihedral Lie algebra of G as a Lie subalgebra of $\text{GrDer}^{SE}L(G)$. We prove the distribution relations in s. 5.7.

The depth 2 and 3 parts of the cohomology of the dihedral Lie algebra of μ_N were related in [G3] to the cohomology of the groups $\Gamma(m; N)$ with coefficients in $S^{w-m}V_m$ for $m = 2, 3$. In the section 6 we compute the cohomology groups

$$H^*(GL_3(\mathbb{Z}), S^{w-3}V_3) \quad \text{and} \quad H^*(\Gamma(3; p), \mathbb{Q})$$

Finally in the chapter 7 we apply the previous results to prove the theorems from the introduction.

3 The setup

1. The action of the Galois group. Recall the map

$$\Phi_N^{(l)} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Aut}\pi_1^{(l)}(X_N, v_\infty) \quad (37)$$

and the projection

$$\pi_1^{(l)}(X_N, v_\infty) \longrightarrow \pi_1^{(l)}(\mathbb{G}_m, v_\infty) = \mathbb{Z}_l(1) \quad (38)$$

Let X be a regular curve over $\overline{\mathbb{Q}}$, \overline{X} the corresponding projective curve and v a tangent vector at $x \in \overline{X}$. Then there is a natural map of Galois modules ([D]):

$$\mathbb{Z}_l(1) = \pi_1^{(l)}(T_x\overline{X} - 0, v) \rightarrow \pi_1^{(l)}(X, v)$$

For $X = \mathbb{P}^1 - \{0, \infty\}$, $x = \infty, v = v_\infty$ it is an isomorphism. So for $X = X_N$ it provides a splitting of (38):

$$I_\infty : \pi_1^{(l)}(\mathbb{G}_m, v_\infty) \hookrightarrow \pi_1^{(l)}(X_N, v_\infty) \quad (39)$$

The subgroup $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty}))$ preserves the elements of the subgroup

$$I_\infty(\mathbb{Z}_l(1)) \subset \pi_1^{(l)}(X_N, v_\infty) \quad (40)$$

Let v, v' be tangent vectors at $x, x' \in \overline{X}$. Denote by $\pi_1^{(l)}(\overline{X}; v, v')$ the pro- l completion of the torsor of path from v to v' . Let $\eta \in \{0\} \cup \mu_N \subset \mathbb{P}^1$. Choose a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$ -invariant tangent vector w_η at the point η . Let p be a pro- l path from v_∞ to w_η . The composition

$$p^{-1} \circ \pi_1^{(l)}(\mathbb{P}^1 - \{0, \eta\}, w_\eta) \circ p$$

provides a map

$$I_\eta(p) : \mathbb{Z}_l(1) \longrightarrow \pi_1^{(l)}(X_N, v_\infty), \quad \eta \in \{0\} \cup \mu_N \quad (41)$$

For a different path p' the map $I_\eta(p')$ is conjugated to $I_\eta(p)$ in $\pi_1^{(l)}$. So the conjugacy class of this map is well defined, hence stable by $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))$.

The restriction of the map $\Phi_N^{(l)}$ to $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ satisfies additional constraints. The group μ_N acts on X_N by $z \mapsto \zeta_N z$. Moreover, there is a natural action of μ_N on $\pi_1^{(l)}(X_N, v_\infty) \otimes \mathbb{Q}_l$ (regarding $\pi_1^{(l)} \otimes \mathbb{Q}_l$ see [D], ch. 9) commuting with the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$, coming as follows. Let $\xi, \zeta, \zeta' \in \mu_N$. The action of ξ on \mathbb{G}_m provides a tangent vector $v_\xi := \xi_* v_\infty$ at ∞ and a map

$$\xi_* : \pi_1^{(l)}(\mathbb{G}_m; v_\zeta, v_{\zeta'}) \longrightarrow \pi_1^{(l)}(\mathbb{G}_m; v_{\xi\zeta}, v_{\xi\zeta'})$$

We have a canonical path $p_{\zeta, \zeta'}$ in $\pi_1^{(l)}(\mathbb{G}_m; v_\zeta, v_{\zeta'}) \otimes \mathbb{Q}_l$ such that

$$p_{\zeta', \zeta''} \circ p_{\zeta, \zeta'} = p_{\zeta, \zeta''}, \quad \xi_* p_{\zeta, \zeta'} = p_{\xi\zeta, \xi\zeta'}$$

Indeed, the composition of path provides an isomorphism of the N -th power of the torsor of path from v_ζ to $v_{\xi\zeta}$ on \mathbb{G}_m with $\pi_1^{(l)}(\mathbb{G}_m)$ which, being abelian, does not depend on the choice of the base point/vector. It is given by

$$\text{com} : p_1 \otimes \dots \otimes p_N \longrightarrow \xi_*^{N-1} p_1 \circ \dots \circ \xi_* p_{N-1} \circ p_N$$

Then

$$p_{\zeta, \xi\zeta} := p \circ \text{com}(p^{\otimes N})^{-1/N} \in \pi_1^{(l)}(\mathbb{G}_m; v_\zeta, v_{\xi\zeta}) \otimes \mathbb{Q}_l$$

Notice that if $(N, l) = 1$ then $p_{\zeta, \zeta'} \in \pi_1^{(l)}(\mathbb{G}_m; v_\zeta, v_{\zeta'})$. There is a natural map

$$\pi_1^{(l)}(\mathbb{P}^1 - \{0, \infty\}; v_\zeta, v_{\zeta'}) \hookrightarrow \pi_1^{(l)}(X_N; v_\zeta, v_{\zeta'})$$

commuting with the action of the Galois group. It provides canonical path

$$p_{\zeta, \zeta'} \in \pi_1^{(l)}(X_N; v_\zeta, v_{\zeta'}) \otimes \mathbb{Q}_l$$

We define the action of an element $\xi \in \mu_N$ on $\alpha \in \pi_1^{(l)}(X_N; v_\infty) \otimes \mathbb{Q}_l$ by $\xi(\alpha) := p_{1, \xi}^{-1} \circ \xi_*(\alpha) \circ p_{1, \xi}$.

So we see that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ preserves the elements of $I_\infty(\mathbb{Z}_l(1))$, the conjugacy classes of the “loops around 0 and ζ_N^a ” provided by (41), and commutes with the action of the group μ_N . Moreover, it is compatible in a natural sense with the maps $X_{MN} \longrightarrow X_M$ given by $z \mapsto z$, $z \mapsto z^N$, see s. 5.7.

2. Passing to the Lie algebras. Let $\mathbb{L}(X_N, v_\infty)$ be the pronilpotent Lie algebra over \mathbb{Q} corresponding by the Maltsev theory to the pronilpotent completion of the fundamental group $\pi_1(X_N(\mathbb{C}), v_\infty)$ (see [D] ch. 9). We use a shorthand \mathbb{L}_N for it. \mathbb{L}_N is isomorphic, not canonically, to the pronilpotent completion of a free Lie algebra with $N+1$ generators corresponding to the loops around 0 and all N -th roots of unity.

Let $\mathbb{L}_N^{(l)}$ be the pronilpotent Lie algebra over \mathbb{Q}_l corresponding to the pro- l group $\pi_1^{(l)}(X_N, v_\infty)$. Namely, set $\pi_1^{(l)} := \pi_1^{(l)}(X_N, v_\infty)$. Recall that $\pi_1^{(l)}(k)$ is the lower central series for the group $\pi_1^{(l)}$. Then $\pi_1^{(l)}/\pi_1^{(l)}(k)$ is an l -adic Lie group, and

$$\mathbb{L}_N^{(l)} := \varprojlim \text{Lie} \left(\pi_1^{(l)}/\pi_1^{(l)}(k) \right) = \mathbb{L}_N \hat{\otimes}_{\mathbb{Q}} \mathbb{Q}_l$$

Similar to (38) and (39) there is a canonical projection

$$p : \mathbb{L}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N), v_\infty) \longrightarrow \mathbb{L}(\mathbb{P}^1 - \{0, \infty\}), v_\infty) \quad (42)$$

and its canonical splitting:

$$i_\infty : \mathbb{L}(\mathbb{P}^1 - \{0, \infty\}, v_\infty) = \mathbb{L}(T_0\mathbb{P}^1, v_\infty) = \mathbb{Q}(1) \hookrightarrow \mathbb{L}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N), v_\infty) \quad (43)$$

Just as in s. 1.2, there are well defined conjugacy classes of “loops around 0 or $\zeta \in \mu_N$ ” based at v_∞ . Set $X_\infty := i_\infty$.

Lemma 3.1 *There exist maps $X_0, X_\zeta : \mathbb{Q}(1) \longrightarrow \mathbb{L}_N$ which belong to the conjugacy classes of the “loops around 0, ζ ” such that $X_0 + \sum_{\zeta \in \mu_N} X_\zeta + X_\infty = 0$ and the action of μ_N permutes X_ζ ’s (i.e. $\xi_* X_\zeta = X_{\xi\zeta}$) and fixes X_0, X_∞ .*

We will use the following notations. Let G be a commutative group written multiplicatively. Let $L(G)$ be the free Lie algebra with the generators X_i where $i \in \{0\} \cup G$ (we assume $0 \notin G$). Set $X_\infty := -X_0 - \sum_{g \in G} X_g$.

Denote by $\mathbb{L}(\mu_N)$ the pronilpotent completion of the Lie algebra $L(\mu_N)$. It is isomorphic (non canonically) to \mathbb{L}_N .

Proof. We follow the argument sketched by the referee. Let \mathbb{H} be the proalgebraic group over \mathbb{Q} consisting of all automorphisms of the Lie algebra $\mathbb{L}(\mu_N)$ which commute with the action of μ_N and preserve the conjugacy classes of the generators X_0, X_ζ, X_∞ . Let \mathbb{T} be the \mathbb{H} -torsor of all maps $X_0, X_\zeta, X_\infty : \mathbb{Q}(1) \longrightarrow \mathbb{L}_N$ satisfying all the conditions of the lemma. Then $\mathbb{T}(\mathbb{C})$ is nonempty. Indeed, in the De Rham realization of \mathbb{L}_N (see [D]) X_ζ, X_∞ are dual to the basis $d \log(z - \zeta), d \log(z)$ in $H_{DR}^1(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N), \mathbb{C})$ and $X_0 := -\sum_{\zeta} X_\zeta - X_\infty$. Applying the comparison theorem between the Betti and De Rham realizations we get an element in $\mathbb{T}(\mathbb{C})$. Since \mathbb{H} is a prounipotent algebraic group one has $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathbb{H}(\overline{\mathbb{Q}})) = 0$, so \mathbb{T} has a \mathbb{Q} -point. The lemma is proved.

There is a homomorphism

$$\varphi_N^{(l)} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Aut} \mathbb{L}_N^{(l)}$$

Let \mathcal{Z}^\bullet be the lower central series for $\mathbb{L}_N^{(l)}$. The quotient $\mathbb{L}_N^{(l)}/\mathcal{Z}^k$ is a finite dimensional Lie algebra over \mathbb{Q}_l . So the image of the subgroup $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ in $\text{Aut}(\mathbb{L}_N^{(l)}/\mathcal{Z}^k)$ is an l -adic Lie group. Denote by $\mathcal{G}_N^{(l)}$ the projective limit (over k) of the Lie algebras of these Lie groups. It is a pro-Lie algebra over \mathbb{Q}_l . It acts by derivations of the Lie algebra $\mathbb{L}_N^{(l)}$. We will describe the constraints on the derivations we get using a more general set up presented below.

3. Special equivariant derivations. A derivation D of the Lie algebra $L(G)$ is called special if there are elements $S_i \in L(G)$ such that

$$D(X_i) = [S_i, X_i] \quad \text{for any } i \in \{0\} \cup G, \quad \text{and} \quad D(X_\infty) = 0 \quad (44)$$

The special derivations of $L(G)$ form a Lie algebra, denoted $\text{Der}^S L(G)$. Indeed, if $D(X_i) = [S_i, X_i]$, $D'(X_i) = [S'_i, X_i]$, then

$$[D, D'](X_i) = [S''_i, X_i], \quad \text{where} \quad S''_i := D(S'_i) - D'(S_i) + [S'_i, S_i] \quad (45)$$

The group G acts on the generators by $h : X_0 \mapsto X_0, X_g \mapsto X_{hg}$. So it acts by automorphisms of the Lie algebra $L(G)$. A derivation D of $L(G)$ is called equivariant if it commutes with the action of G . Let $\text{Der}^{SE} L(G)$ be the Lie algebra of all special equivariant derivations of the Lie algebra $L(G)$.

4. The weight and depth filtration on \mathbb{L}_N . There are two increasing filtrations by ideals on the Lie algebra \mathbb{L}_N , indexed by integers $n \leq 0$.

The weight filtration \mathcal{F}_\bullet^W . It coincides with the lower central series for \mathbb{L}_N :

$$\mathbb{L}_N = \mathcal{F}_{-1}^W \mathbb{L}_N; \quad \mathcal{F}_{-n-1}^W \mathbb{L}_N := [\mathcal{F}_{-n}^W \mathbb{L}_N, \mathbb{L}_N]$$

The depth filtration \mathcal{F}_\bullet^D . Let \mathcal{I}_N be the kernel of projection (42). Its powers give the depth filtration:

$$\mathcal{F}_0^D \mathbb{L}_N = \mathbb{L}_N, \quad \mathcal{F}_{-1}^D \mathbb{L}_N = \mathcal{I}_N, \quad \mathcal{F}_{-n-1}^D \mathbb{L}_N = [\mathcal{I}_N, \mathcal{F}_{-n}^D \mathbb{L}_N]$$

The weight filtration can be defined on the Lie algebra corresponding to the pronilpotent completion of the fundamental group of an arbitrary algebraic variety. The weights on \mathbb{L}_N are obtained by dividing by 2 the usual weights. The weight filtration admits a splitting, i.e. it is defined by a grading.

These filtrations induce two filtrations on the Lie algebra $\text{Der}^{SE} \mathbb{L}_N$. Taking the associated graded with respect to these filtrations we get a Lie algebra $\text{GrDer}_{\bullet\bullet}^{SE} \mathbb{L}_N$ bigraded by the weight $-w$ and depth $-m$.

5. The weight grading and depth filtration on $\text{Der}^S L(G)$. The Lie algebra $L(G)$ is bigraded by the weight and depth. Namely, the free generators X_0, X_g are bihomogeneous: they are of weight -1 , X_0 is of depth 0 and the X_g 's are of depth -1 .

Each of the gradings induces a filtration of $L(G)$. The weight filtration is given by the lower central series. It goes from $-\infty$ to -1 . Let \mathcal{I} be the kernel of the natural projection $L(G) \rightarrow \mathbb{Q}$ given by $X_g \mapsto 0, X_0 \mapsto 1$. Its powers provide the depth filtration on $L(G)$. It goes from $-\infty$ to 0.

The Lie algebra $\text{Der} L(G)$ is bigraded by the weight and depth. Its Lie subalgebras $\text{Der}^S L(G)$ and $\text{Der}^{SE} L(G)$ are compatible with the weight grading. However they are *not* compatible with the depth grading. Therefore they are graded by the weight, and *filtered* by the depth. A derivation (44) is of depth $-m$ if each $S_j \bmod X_j$ is of depth $-m$, i.e. there are at least m X_i 's different from X_0 in $S_j \bmod X_j$. The depth filtration is compatible with the weight grading. Let $\text{GrDer}_{\bullet\bullet}^{SE} L(G)$ be the associated graded for the depth filtration.

One of our key tools is the following theorem proved in s. 5.

Theorem 3.2 *Let G be a finite commutative group. Then there exists an injective morphism of bigraded Lie algebras*

$$\xi_G : D_{\bullet\bullet}(G) \hookrightarrow \mathrm{GrDer}_{\bullet\bullet}^{SE} L(G)$$

6. The problem of describing of the map $\varphi_N^{(l)}$. There is no canonical choice of the generators X_0, X_ζ of the Lie algebra \mathbb{L}_N satisfying all the conditions of lemma 3.1. However their projections in $\mathbb{L}_N/[\mathbb{L}_N, \mathbb{L}_N]$ are independent of the choice involved. So we have a *canonical* isomorphism

$$i : L(\mu_N) \xrightarrow{=} \mathrm{Gr}_{\bullet}^W \mathbb{L}_N \quad (46)$$

It preserves the depth filtration.

Remark. Let $\mathbb{L}(X, x)$ be the fundamental Lie algebra of X based at a point/tangent vector x . If y is another base point/vector then there is a torsor of isomorphisms $\mathbb{L}(X_N, x) \longrightarrow \mathbb{L}(X_N, y)$ defined up to a conjugation. However the isomorphism of the associated graded for the lower series filtration L_* is a *canonical* isomorphism

$$i_{x,y} : \mathrm{Gr}^L \mathbb{L}(X, x) \longrightarrow \mathrm{Gr}^L \mathbb{L}(X, y)$$

If $X = X_N$ then the lower series filtration coincides with the weight filtration. Thus the graded Lie algebra $\mathrm{Gr}^W \mathbb{L}_N := \mathrm{Gr}^W \mathbb{L}(X_N, v_\infty)$ does not depend on the choice of the base point or tangent vector v_∞ . So (46) is indeed a canonical isomorphism.

It follows from s. 3.2 that $\mathcal{G}_N^{(l)}$ acts by special equivariant derivations of the Lie algebra $\mathbb{L}_N^{(l)}$, i.e.

$$\mathcal{G}_N^{(l)} \subset \mathrm{Der}^{SE} \mathbb{L}_N^{(l)} \quad (47)$$

The map $\varphi_N^{(l)}$ obviously respects the two filtrations. The filtrations on $\mathrm{Der}^{SE} \mathbb{L}_N^{(l)}$ induce two filtrations on $\mathcal{G}_N^{(l)}$. Let $\mathrm{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$ be the associated graded of the Lie algebra $\mathcal{G}_N^{(l)}$. There is an inclusion $\mathrm{Gr} \mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \subset \mathrm{GrDer}_{\bullet\bullet}^{SE} \mathbb{L}_N^{(l)}$ which, as was stressed by the referee, is provided by the fact that the weight filtration admits a splitting, i.e. is defined by a grading, and both the depth filtration and the subspace (47) are compatible with this grading. Such a weight splitting is provided by the eigenspaces of a Frobenius F_p , $p \nmid N$. Notice that F_p normalizes the normal subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$.

Let $W_{\bullet} L$ be a filtration on a vector space L . A splitting $\varphi : \mathrm{Gr}^W L \longrightarrow L$ of the filtration leads to an isomorphism $\varphi^* : \mathrm{End}(L) \longrightarrow \mathrm{End}(\mathrm{Gr}^W L)$. The space $\mathrm{End}(L)$ inherits a natural filtration, while $\mathrm{End}(\mathrm{Gr}^W L)$ is graded. The map φ^* respects the corresponding filtrations. The map

$$\mathrm{Gr} \varphi^* : \mathrm{Gr}^W(\mathrm{End} L) \longrightarrow \mathrm{End}(\mathrm{Gr}^W L)$$

does not depend on the choice of the splitting. Applying this to the case $L = \mathbb{L}_N$ we get a *canonical* isomorphism

$$\mathrm{Gr}^W(\mathrm{Der}^{SE}\mathbb{L}_N) \cong \mathrm{Der}^{SE}L(\mu_N) \quad (48)$$

respecting the weight grading. Thus there is a *canonical* injective morphism

$$\mathrm{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N) \hookrightarrow \mathrm{GrDer}_{\bullet\bullet}^{SE}\mathbb{L}_N^{(l)} \cong \mathrm{GrDer}_{\bullet\bullet}^{SE}L(\mu_N) \otimes \mathbb{Q}_l \quad (49)$$

The goal of this paper is to study Lie subalgebra (49). It is canonically isomorphic to the Galois Lie algebra introduced in section 1. Indeed, the Lie algebras of images if the following two maps coincide:

$$\begin{aligned} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) &\longrightarrow \mathrm{Aut}(\pi_1^{(l)}(X_N, v_\infty)_{[-w, -m]}) \\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) &\longrightarrow \mathrm{Aut}\mathbb{L}_N^{(l)}/(\mathcal{F}_{-w-1}^W + \mathcal{F}_{-m-1}^D) \end{aligned}$$

Why did we take the associated graded of $\mathcal{G}_N^{(l)}$ for the weight and depth filtrations? The Lie algebra $\mathcal{G}_N^{(l)}$ is isomorphic to $\mathrm{Gr}_{\bullet}^W\mathcal{G}_N^{(l)}$, but this isomorphism is not canonical. $\mathrm{Gr}_{\bullet}^W\mathcal{G}_N^{(l)}$ is a Lie subalgebra of $\mathrm{Gr}^W(\mathrm{Der}^{SE}\mathbb{L}_N) \otimes \mathbb{Q}_l$. Via canonical isomorphism (48) it became a Lie subalgebra of $\mathrm{Der}^{SE}L(\mu_N) \otimes \mathbb{Q}_l$, which has natural generators provided by the canonical generators of $L(\mu_N)$ (see s. 5.2). This gives canonical “coordinates” for description of elements of $\mathrm{Gr}_{\bullet}^W\mathcal{G}_N^{(l)}$. Another reason to consider the Lie algebra $\mathrm{Gr}_{\bullet}^W\mathcal{G}_N^{(l)}$ is provided by its motivic interpretation, see s. 3.7 below. The surprising benefit of taking its associated graded for the *depth* filtration is an unexpected relation with the geometry of modular varieties for GL_m , where m is the depth.

Remark. Usually the associated graded for the depth filtration is not isomorphic to a subalgebra of a Lie algebra of any quotient of the Galois group. The situation is different in the following two cases when the associated graded for the depth filtration is isomorphic to the original Lie algebra:

i) *In the depth two case* $\mathrm{Gr}\mathcal{G}_{\bullet, \geq -2}^{(l)}(\mu_N)$ is isomorphic to $\mathrm{Gr}^W\mathcal{G}_N^{(l)}/\mathcal{F}_{-3}^D\mathrm{Gr}^W\mathcal{G}_N^{(l)}$ because there is no room for the difference.

ii) *In the diagonal case*, as we noticed in s. 2.7, the diagonal Lie algebra defined as a Lie subalgebra of $\mathrm{Gr}^W\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$, is non canonically isomorphic to a Lie subalgebra of $\mathrm{Gr}^W\mathcal{G}_N^{(l)}$.

7. Motivic interpretation of the Lie algebra $\mathrm{Gr}_{\bullet}^W\mathcal{G}_N^{(l)}$. Let me recall the mixed Tate category of lisse l -adic sheaves defined by Beilinson and Deligne, following closely to the first 4 pages of chapter 1 of [BD]. For a connected coherent scheme S over $\mathbb{Z}[1/l]$ such that $\mu_{l^\infty} \not\subset \mathcal{O}^*(S)$ denote by $\mathcal{F}_{\mathbb{Q}_l}(S)$ the Tannakian category of lisse \mathbb{Q}_l -sheaves on S . There are the Tate sheaves $\mathbb{Q}_l(m) := \mathbb{Q}_l(m)_S := \mathbb{Q}_l(1)_S^{\otimes m}$. Thanks to our condition they are mutually non-isomorphic. Call an object of $\mathcal{F}_{\mathbb{Q}_l}(S)$ a mixed Tate object if it admits a finite increasing filtration W , indexed by \mathbb{Z} , such that Gr_k^W is a direct sum of copies of

$\mathbb{Q}_l(-k)$. Let $\mathcal{TF}_{\mathbb{Q}_l}(S)$ be the full subcategory of mixed Tate objects in $\mathcal{F}_{\mathbb{Q}_l}(S)$. Then it is a Tannakian \mathbb{Q}_l -category, and obviously one has

$$\mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = 0 \quad \text{for } m \leq 0$$

(This may not be so for the Ext^1 in the category $\mathcal{F}_{\mathbb{Q}_l}(S)$). So $\mathcal{TF}_{\mathbb{Q}_l}(S)$ is a mixed Tate category in the terminology of [BD]. It is easy to deduce from this that any its object admits a unique filtration W such that Gr_k^W is a direct sum of copies of $\mathbb{Q}_l(-k)$, and any morphism is strictly compatible with W . There is canonical functor ω to the category of graded \mathbb{Q}_l -vector spaces:

$$\omega : X \longmapsto \bigoplus_m \mathrm{Hom}(\mathbb{Q}_l(m), \mathrm{Gr}_{-m}^W X)$$

It is an exact functor commuting with \otimes -product. It is called canonical fiber functor. Let $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ be the Lie algebra of all \otimes -derivations of ω . By definition a degree i element $\alpha \in L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}_i$ is a collection of natural transformation $\alpha_j : \omega_j \longrightarrow \omega_{j+i}$ such that $\alpha_{M \otimes N} = \alpha_M \otimes \mathrm{Id}_{\omega(N)} + \mathrm{Id}_{\omega(M)} \otimes \alpha_N$ and $\alpha_{jM(1)} = \alpha_{(j+1)M}$. Then $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ is graded pro-Lie algebra over \mathbb{Q}_l . It is called the fundamental Lie algebra of the mixed Tate category $\mathcal{TF}_{\mathbb{Q}_l}(S)$. The fiber functor provides an equivalence between the category $\mathcal{TF}_{\mathbb{Q}_l}(S)$ and the category of finite dimensional graded $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ -modules.

The standard Tannakian formalism in this case works as follows. Forgetting the graded structure on $\omega(X)$ we get a fiber functor $\tilde{\omega}$. Its \otimes -automorphisms provide a group scheme over \mathbb{Q}_l which is a direct product of \mathbb{G}_m and a prounipotent group. Then $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ is the Lie algebra of the prounipotent part. The action of \mathbb{G}_m provides a grading on it.

From now on let S be the spectrum of the ring of integers of a number field F punctured in a finite set \mathcal{S} containing all primes above l . Then $\mathcal{F}_{\mathbb{Q}_l}(S)$ is identified with the category of finite dimensional l -adic representations of $\mathrm{Gal}(\overline{F}/F)$ unramified outside of \mathcal{S} . The underlying vector space of a Galois representation provides a fiber functor on $\mathcal{F}_{\mathbb{Q}_l}(S)$, and thus another fiber functor on the category $\mathcal{TF}_{\mathbb{Q}_l}(S)$. It follows that for any Galois representation V from the category $\mathcal{TF}_{\mathbb{Q}_l}(S)$ the Lie algebra of Zariski closure of the image of the Galois group in $\mathrm{Aut} V$ is isomorphic to the image of the semidirect product of $\mathrm{Lie} \mathbb{G}_m$ and $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ acting on $\tilde{\omega}(V)$. Moreover, let \mathcal{G}_V be Zariski closure of the image of $\mathrm{Gal}(\overline{F}/F(\zeta_{l^\infty}))$ in $\mathrm{Aut} V$. Then the Lie algebra \mathcal{G}_V is isomorphic to the image of $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ in $\mathrm{Der} \omega(V)$. This isomorphism is canonical for $\mathrm{Gr}^W \mathcal{G}_V$.

As noticed in s. 1.2.3 of [BD], it follows from a theorem of Soule [So] that the l -adic regulator map provides canonical isomorphism

$$H_{\mathcal{M}}^i(S, \mathbb{Q}(m)) \otimes \mathbb{Q}_l := gr_m^\gamma K_{2m-i}(S) \otimes \mathbb{Q}_l \xrightarrow{=} \mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^i(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) \quad (50)$$

Indeed, let $G_{F_{\mathcal{S}}}$ be the Galois group of the maximal extension of F unramified outside \mathcal{S} . Soule proved that the l -adic regulator map provides isomorphism

$$H_{\mathcal{M}}^i(S, \mathbb{Q}(m)) \otimes \mathbb{Q}_l \xrightarrow{=} H^i(G_{F_{\mathcal{S}}}, \mathbb{Q}_l(m))$$

for $i = 1, m \geq 1$, and also for $i = 2, m \geq 2$ where both groups are zero: for the left one this follows from Borel's theorem; the $l = 2$ case see s. B4 of [HW].

By the very definitions for $m \geq 0$

$$\mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = \mathrm{Ext}_{\mathcal{F}_{\mathbb{Q}_l}(S)}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = H^1(G_{F_S}, \mathbb{Q}_l(m)),$$

so (50) follows for $i = 1$ case. Further, for $m \geq 2$

$$\mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^2(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) \subset \mathrm{Ext}_{\mathcal{F}_{\mathbb{Q}_l}(S)}^2(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = H^2(G_{F_S}, \mathbb{Q}_l(m)) = 0 \quad (51)$$

To check the first inclusion notice the following. In any abelian category every element in the Yoneda Ext^2 is a product of Ext^1 's. The Yoneda product of extension classes $\alpha \in \mathrm{Ext}^1(A, B)$ and $\beta \in \mathrm{Ext}^1(B, C)$ is zero in $\mathrm{Ext}^2(A, C)$ if and only if there exists an object M with a three step filtration F such that $F_0 M = A, F_1 M = \alpha$ and $M/F_0 M = \beta$. In our case we may assume that a class in $\mathrm{Ext}^2(\mathbb{Q}_l(0), \mathbb{Q}_l(m))$ is a product the classes in $\mathrm{Ext}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(a))$ and $\mathrm{Ext}^1(\mathbb{Q}_l(a), \mathbb{Q}_l(m))$. Their product is zero in $\mathrm{Ext}_{\mathcal{F}_{\mathbb{Q}_l}(S)}^2$, and filtration F on the corresponding object M can be used as a weight filtration, so M belongs to the category $\mathcal{TF}_{\mathbb{Q}_l}(S)$. Finally, $\mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^2(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = 0$ for $m < 2$ since, as before, we can decompose it in a product of $\mathrm{Ext}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(k))$'s one of whom must have $k \leq 0$ and thus be zero. Therefore

$$\mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^2(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) = 0 \quad \text{for all } m$$

This just means that the fundamental Lie algebra $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$ is a free negatively graded Lie algebra generated in the degree $-m$ by the dual to \mathbb{Q}_l -vector space $K_{2m-1}(S) \otimes \mathbb{Q}_l$.

Now let $\mathcal{S}' := \mathcal{S} - \{l\}$ and S' be the spectrum of the ring of \mathcal{S}' -integers, then the l -adic regulator map

$$H_{\mathcal{M}}^i(S', \mathbb{Q}(m)) \otimes \mathbb{Q}_l \longrightarrow \mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(m)) \quad (52)$$

is still isomorphism $i = 1, 2$ and $m \geq 2$ since each of the groups does not change when we delete a closed point from the spectrum. If $i = 1, m = 1$ map (52) is injective but not an isomorphism since

$$H_{\mathcal{M}}^1(S', \mathbb{Q}(1)) \otimes \mathbb{Q}_l = \mathcal{O}_{S'}^* \otimes \mathbb{Q}_l \hookrightarrow \mathrm{Ext}_{\mathcal{TF}_{\mathbb{Q}_l}(S)}^1(\mathbb{Q}_l(0), \mathbb{Q}_l(1)) = E_S \otimes \mathbb{Q}_l$$

where E_S is the group of S -units. Thus $L_{\mathcal{TF}_{\mathbb{Q}_l}(S')}$ is a quotient of $L_{\mathcal{TF}_{\mathbb{Q}_l}(S)}$, and the space of generators of these Lie algebras differ only in the degree -1 .

Since $G_m - \mu_N$ has a good reduction outside of N , in our case $S = S_N \cup \{l\}$, and the Lie algebra \mathbb{L}_N is a pro-object in $\mathcal{TF}_{\mathbb{Q}_l}(S_N \cup \{l\})$. So the Lie algebra $\mathrm{Gr}_{\bullet}^W \mathcal{G}_N^{(l)}$ is canonically isomorphic to the image of the Lie algebra $L_{\mathcal{TF}_{\mathbb{Q}_l}(S_N \cup \{l\})}$ acting by derivations of the pro-object $\Psi(\mathbb{L}_N)$. However it follows from the distribution relations (see s. 5.7) that the degree -1 component of Lie algebra

$L_{\mathcal{T}\mathbb{Q}_l}(S_N \cup \{l\})$ always act through its motivic quotient dual to the subspace $\mathcal{O}_{S'}^* \otimes \mathbb{Q}_l$.

Recently some of the arguments similar to the one of Beilinson and Deligne described above were rediscovered by Hain and Matsumoto [HM].

Moreover let $L_{\mathcal{T}\mathcal{M}}(S)$ be the fundamental Lie algebra of the mixed Tate category of mixed Tate motives over S ([G4]). Then $L_{\mathcal{T}\mathbb{Q}_l}(S) = L_{\mathcal{T}\mathcal{M}}(S) \otimes \mathbb{Q}_l$. We will not use this in the present paper. See also s. 3 of [G8] for the formalism of motivic Lie algebras in the general setup.

Deligne's arguments show that (assuming the motivic formalism) the Lie algebra $\mathrm{Gr}_{\bullet}^W \mathcal{G}_N^{(l)}$ is free for $N = 2$ ([D2]) and $N = 3, 4$ ([D3]). Corollary 2.16 implies that it is not free for sufficiently big N , for instance for prime $p > 5$: the modular forms on $\Gamma_1(p)$ provide an obstruction to freeness.

4 The dihedral Lie coalgebra

1. Definitions. Let G and H be two commutative groups, or more generally two commutative group schemes. Then, generalizing a construction given in [G3], one can define a graded Lie coalgebra $\mathcal{D}_{\bullet}(G|H)$, called the dihedral Lie algebra of G and H (see [G4]). In the special case when $H = \mathrm{Spec}\mathbb{Q}[[t]]$ is the additive group of the formal line it essentially coincides with a bigraded Lie coalgebra $\mathcal{D}_{\bullet\bullet}(G)$ defined in [G3] and called the dihedral Lie algebra of G . (The second grading is related to the natural filtration on $\mathbb{Q}[[t]]$). The construction of $\mathcal{D}_{\bullet}(G|H)$ is left as an easy exercise, see the end of s. 4.5.

Below we give a version of the definition of the bigraded Lie coalgebra

$$\mathcal{D}_{\bullet\bullet}(G) = \oplus_{w \geq m \geq 1} \mathcal{D}_{w,m}(G)$$

given in [G3]. We will use it only when $G = \mu_N$.

The \mathbb{Q} -vector space $\mathcal{D}_{w,m}(G)$ is generated by the symbols

$$I_{n_1, \dots, n_m}(g_1 : \dots : g_{m+1}), \quad w = n_1 + \dots + n_m, \quad n_i \geq 1 \quad (53)$$

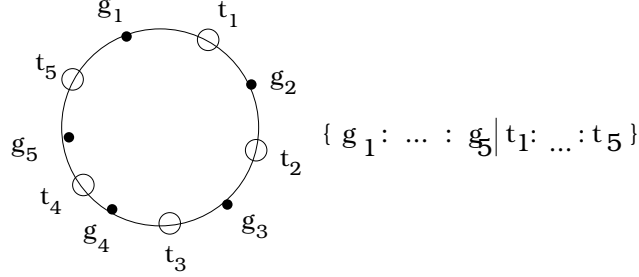
To define the relations we introduce the generating series

$$\begin{aligned} \{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} := \\ \sum_{n_i > 0} I_{n_1, \dots, n_m}(g_1 : \dots : g_{m+1})(t_1 - t_{m+1})^{n_1-1} \dots (t_m - t_{m+1})^{n_m-1} \end{aligned} \quad (54)$$

Thus

$$\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} = \{g_1 : \dots : g_{m+1} | t + t_1 : \dots : t + t_{m+1}\} \quad (55)$$

We think about the generating series (54) as a function of $m+1$ pairs of variables $(g_1, t_1), \dots, (g_{m+1}, t_{m+1})$ located cyclically on an oriented circle (and called a dihedral word) as follows. The oriented circle has slots, where the g 's sit, and in between the consecutive slots, dual slots, where t 's sit, see the picture below. The slots are marked by black points, and the dual slots by little circles.



We will also need two other generating series:

$$\{g_1 : \dots : g_{m+1} | t_1, \dots, t_{m+1}\} := \quad (56)$$

$$\{g_1 : \dots : g_{m+1} | t_1 : t_1 + t_2 : \dots : t_1 + \dots + t_m : 0\}$$

where $t_1 + \dots + t_{m+1} = 0$, and

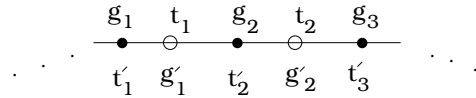
$$\{g_1, \dots, g_{m+1} | t_1 : \dots : t_{m+1}\} := \{1 : g_1 : g_1 g_2 : \dots : g_1 \dots g_m | t_1 : \dots : t_{m+1}\} \quad (57)$$

where $g_1 \cdot \dots \cdot g_{m+1} = 1$.

To make these definitions more transparent set

$$g'_i := g_i^{-1} g_{i+1}, \quad t'_i := -t_{i-1} + t_i$$

and put them on the circle together with g 's and t 's as follows



Then it is easy to check that

$$\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} = \{g_1 : \dots : g_{m+1} | t'_1, \dots, t'_{m+1}\} = \quad (58)$$

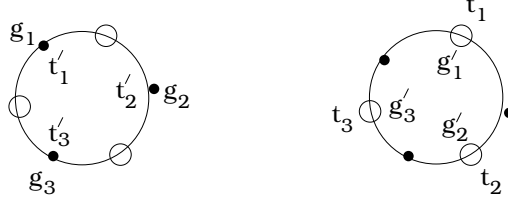
$$\{g'_1, \dots, g'_{m+1} | t_1 : \dots : t_{m+1}\}$$

So we have

$$\{g_1 : \dots : g_{m+1} | t_1, \dots, t_{m+1}\} = \{g'_1, \dots, g'_{m+1} | t_1 : t_1 + t_2 : \dots : t_1 + \dots + t_m : 0\} \quad (59)$$

$$\{g_1, \dots, g_{m+1} | t_1 : \dots : t_{m+1}\} = \{1 : g_1 : g_1 g_2 : \dots : g_1 \dots g_m | t'_1, t'_2, \dots, t'_{m+1}\} \quad (60)$$

We picture the three generating series (58) on an oriented circle. Namely, for each of them we leave on the circle only the two sets of variables among g 's, g' 's, t 's, t' 's which appear in this generating series, see the picture.



$$\{g_1 : g_2 : g_3 \mid t'_1, t'_2, t'_3\} \quad \{g'_1, g'_2, g'_3 \mid t_1 : t_2 : t_3\}$$

The “ $\{:\}$ ”-variables are outside, and the “ $\{,\}$ ”-variables are inside of the circle. (One can have the fourth type of the generating series, but it plays no role in our story). For the $\{g : |t, \}$ -generating series the variables sit only at slots, and for the $\{g, |t : \}$ -generating series only at dual slots.

Relations. i) *Homogeneity.* For any $g \in G$ one has

$$\{g \cdot g_1 : \dots : g \cdot g_{m+1} \mid t_1 : \dots : t_{m+1}\} = \{g_1 : \dots : g_{m+1} \mid t_1 : \dots : t_{m+1}\} \quad (61)$$

(A similar relation for t 's, when $t_i \mapsto t_i + t$, is true by the definition (54)).

ii) *The double shuffle relations* ($p + q = m, p \geq 1, q \geq 1$).

$$s_1^{p,q}(g_1, \dots, g_m, g_{m+1} \mid t_1 : \dots : t_m : t_{m+1}) :=$$

$$\sum_{\sigma \in \Sigma_{p,q}} \{g_{\sigma(1)}, \dots, g_{\sigma(m)}, g_{m+1} \mid t_{\sigma(1)} : \dots : t_{\sigma(m)} : t_{m+1}\} = 0 \quad (62)$$

$$s_2^{p,q}(g_1 : \dots : g_m : g_{m+1} \mid t_1, \dots, t_m, t_{m+1}) :=$$

$$\sum_{\sigma \in \Sigma_{p,q}} \{g_{\sigma(1)} : \dots : g_{\sigma(m)} : g_{m+1} \mid t_{\sigma(1)}, \dots, t_{\sigma(m)}, t_{m+1}\} = 0 \quad (63)$$

iii) *The distribution relations.* Let $l \in \mathbb{Z}$. Suppose that the l -torsion subgroup G_l of G is finite and its order is divisible by l . Then if x_1, \dots, x_m are l -powers

$$\{x_1 : \dots : x_{m+1} \mid t_1 : \dots : t_{m+1}\} - \frac{1}{|G_l|} \sum_{y_i' = x_i} \{y_1 : \dots : y_{m+1} \mid l \cdot t_1 : \dots : l \cdot t_{m+1}\} = 0$$

except that a constant (in t) is allowed when $m = 1$ and $x_1 = x_2$, so that $I_1(e : e) - \sum_{y' = e} I_1(y : e)$ is not necessarily zero.

iv) $I_1(e : e) = 0$. (In [G3] we did not impose this condition).

Denote by $\widehat{\mathcal{D}}_{\bullet\bullet}(G)$ the bigraded vector space defined by the conditions i)-iii) only. Then $\widehat{\mathcal{D}}_{\bullet\bullet}(G) = \mathcal{D}_{\bullet\bullet}(G) \oplus \mathbb{Q}_{(1,1)}$ where $\mathbb{Q}_{(1,1)}$ is generated by $I_1(e : e)$.

Remark. If $m \geq 2$, the relation (63) for $g_1, \dots, g_{m+1} = e$ implies that $I_{1,\dots,1}(e : \dots : e) = 0$, and iv) requires this vanishing to hold for $m = 1$ as well.

Remark. To define a map from $\mathcal{D}_{\bullet\bullet}(G)$ to a vector space V amounts to define generating series $\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\}$ with coefficients in V , that is, in the $V[[t_1, \dots, t_{m+1}]]$, obeying (55) and i) - v).

Remark. Altering one or both of the conditions iii) and iv) we can still get important Lie coalgebras.

2. The dihedral symmetry relations. By definition they consist of the following list of relations:

The cyclic symmetry relations:

$$\{g_1 : g_2 : \dots : g_{m+1} | t_1 : t_2 : \dots : t_{m+1}\} = \{g_2 : \dots : g_{m+1} : g_1 | t_2 : \dots : t_{m+1} : t_1\} \quad (64)$$

The reflection relation:

$$\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} = (-1)^{m+1} \{g_{m+1} : \dots : g_1 | -t_m : \dots : -t_1 : -t_{m+1}\} \quad (65)$$

The inversion relations:

$$\{x_1^{-1} : \dots : x_{m+1}^{-1} | t_1 : t_2 : \dots : t_{m+1}\} = \{x_1 : \dots : x_{m+1} | -t_1 : \dots : -t_{m+1}\} \quad (66)$$

The inversion relations are precisely the distribution relations for $l = -1$.

One can check that the reflection relations (65) for the $\{a, |b : \}$ -generating series look as follows:

$$\{g_1, \dots, g_{m+1} | t_1 : \dots : t_{m+1}\} = (-1)^{m+1} \{g_{m+1}, \dots, g_1 | t_{m+1} : \dots : t_1\} \quad (67)$$

Using (66), one checks the same identity for the $\{a : |b, \}$ -generating series. This tells the effect of changing the orientation of the circle.

The dihedral symmetry for $m = 1$ reduces to the inversion relation.

Theorem 4.1 *The double shuffle relations in the case $m \geq 2$ imply the dihedral symmetry relations.*

Proof. One has

$$\sum_{k=0}^m (-1)^k s_2^{k, m-k} (g_k : g_{k-1} : \dots : g_1 : g_{k+1} : \dots : g_m : g_{m+1} |$$

$$t_k, t_{k-1}, \dots, t_1, t_{k+1}, \dots, t_m, t_{m+1}) = 0$$

Indeed, let us decompose the set of all shuffles of the ordered sets $\{k, k-1, \dots, 1\}$ and $\{k+1, k+2, \dots, m\}$ into a union of two subsets, denoted S'_k and S''_k . The subset S''_k consists of those shuffles where $k+1$ appears on the left of k . After the summation with signs the terms corresponding to S''_k cancel the ones corresponding to S'_{k+1} . This means that

$$\{g_1 : g_2 : \dots : g_m : g_{m+1} | t_1, t_2, \dots, t_m, t_{m+1}\} = \quad (68)$$

$$(-1)^{m+1}\{g_m : g_{m-1} : \dots : g_1 : g_{m+1} | t_m, t_{m-1}, \dots, t_1, t_{m+1}\}$$

Similarly

$$\begin{aligned} & \{g_1, g_2, \dots, g_m, g_{m+1} | t_1 : t_2 : \dots : t_m : t_{m+1}\} = \\ & (-1)^{m+1}\{g_m, g_{m-1}, \dots, g_1, g_{m+1} | t_m : t_{m-1} : \dots : t_1 : t_{m+1}\} \end{aligned} \quad (69)$$

To simplify the formulas below we use notation $h_i := g'_i = g_i^{-1}g_{i+1}$. Using the identities we just get we have

$$\begin{aligned} & \{h_1, \dots, h_{m+1} | t_1 : \dots : t_{m+1}\} \stackrel{(58)}{=} \{g_1 : \dots : g_{m+1} | t'_1, \dots, t'_{m+1}\} \stackrel{(68)}{=} \\ & (-1)^{m+1}\{g_m : g_{m-1} : \dots : g_1 : g_{m+1} | t'_m, t'_{m-1}, \dots, t'_1, t'_{m+1}\} \\ & \stackrel{(59)}{=} (-1)^{m+1}\{h_{m-1}^{-1}, h_{m-2}^{-1}, \dots, h_1^{-1}, h_{m+1}^{-1}, h_m^{-1} | \\ & t_m - t_{m-1} : t_m - t_{m-2} : \dots : t_m - t_1 : t_m - t_{m+1} : 0\} \stackrel{(69)}{=} \\ & \{h_{m+1}^{-1}, h_1^{-1}, h_2^{-1}, \dots, h_{m-1}^{-1}, h_m^{-1} | -t_{m+1} : -t_1 : -t_2 : \dots : -t_{m-1} : -t_m\} \end{aligned}$$

Therefore we get

$$\begin{aligned} & \{g_1, g_2, \dots, g_{m+1} | t_1 : t_2 : \dots : t_{m+1}\} = \\ & \{g_{m+1}^{-1}, g_1^{-1}, g_2^{-1}, \dots, g_m^{-1} | -t_{m+1} : -t_1 : -t_2 : \dots : -t_m\} \end{aligned} \quad (70)$$

It is not difficult to check that

$$\{g_1, \dots, g_{m+1} | t_1 : \dots : t_{m+1}\} + \quad (71)$$

$$\begin{aligned} & s_1^{m-1,1}(g_1, \dots, g_{m-1}, g_{m+1}, g_m | t_1 : \dots : t_{m-1} : t_{m+1} : t_m) - \\ & \{g_{m+1}, g_1, \dots, g_m | t_{m+1} : t_1 : \dots : t_m\} = \end{aligned} \quad (72)$$

$$\sum \{g_1, g_{\sigma(2)}, \dots, g_{\sigma(m+1)} | t_1 : t_{\sigma(2)} : \dots : t_{\sigma(m+1)}\} \quad (73)$$

where the sum is over all shuffles of the sets $\{2, 3, \dots, m\}$ and $\{m+1\}$. Expression (73) does not have the form of a shuffle relation. However applying (70) we see that it is equal to a shuffle relation. Therefore relation (71) = (72) provides the cyclic symmetry for the generators (57). Applying (70) we get inversion relations (66) for these generators. Finally, using (69) and the cyclic relations we get reflection relation (67) for them. So we proved the dihedral symmetry for the generators (57), and thus for the generators (56) and (54). Theorem 4.1 is proved.

Proof of theorem 2.7. It is identical to the proof of theorem 4.1 after we suppress t 's, and rename g 's by v 's, and use the additive notations instead of the multiplicative.

Corollary 4.2 *If G is a trivial group then $\mathcal{D}_{w,m} = 0$ if $w + m$ is odd.*

Proof. One has $\{e : \dots : e | t_1, \dots, t_m, 0\} = \{e : \dots : e | -t_1, \dots, -t_m, 0\}$ by the inversion relation. This immediately leads to the statement.

3. Motivation: relation with multiple polylogarithms when $G = \mu_N$. There are several natural sets of the generators of the vector spaces $\mathcal{D}_{w,m}(G)$ which are reminiscent of different definitions of multiple polylogarithms. It is useful to keep them in mind in order to trace the origins of the definitions given above.

i) Let $x_1 \dots x_{m+1} = 1$. We define the generators $L_{n_1, \dots, n_m}(x_1, \dots, x_m)$ as the coefficients of the generating series (57) when $t_{m+1} := 0$:

$$\{x_1, \dots, x_{m+1} | t_1 : \dots : t_m : 0\} =: \sum_{n_i > 0} L_{n_1, \dots, n_m}(x_1, \dots, x_m) t_1^{n_1-1} \dots t_m^{n_m-1}$$

If $G = \mu_N$ they are related to multiple polylogarithms

$$Li_{n_1, \dots, n_m}(x_1, \dots, x_m) := \sum_{0 < k_1 < k_2 < \dots < k_m} \frac{x_1^{k_1} x_2^{k_2} \dots x_m^{k_m}}{k_1^{n_1} k_2^{n_2} \dots k_m^{n_m}} \quad (74)$$

considered modulo the products of similar power series and modulo the lower depth power series. The shuffle relations (62) correspond precisely to the shuffle product formula ([G3]) for this power series.

The distribution relations are easy to check for the power series (74).

ii) The generators $I_{n_1, \dots, n_m}(a_1 : a_2 : \dots : a_{m+1})$ are the coefficients of the generating series (54) when $t_{m+1} = 0$:

$$\{a_1 : \dots : a_{m+1} | t_1 : \dots : t_m : 0\} =: \sum_{n_i > 0} I_{n_1, \dots, n_m}(a_1 : \dots : a_{m+1}) t_1^{n_1-1} \dots t_m^{n_m-1}$$

When $G = \mu_N$ they are related to the iterated integrals

$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{a_1 - t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_1} \circ \dots \circ \underbrace{\frac{dt}{a_m - t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_m} \quad (75)$$

considered modulo the products of similar integrals and modulo the lower depth integrals. The shuffle relations (63) correspond to the shuffle product formula for the iterated integrals (75), see theorem 2.2 in [G3] and more details in [G4]. Formula (57) reflects theorem 2.1 in [G3] relating power series (74) with iterated integrals (75). Formula (56) reflects the definition of the generating series I^* which appear in theorem 2.2 in [G3].

Remark. There is a striking symmetry between g 's and t 's in the generating series (54) or (56) and (57). It is completely unexpected even when $G = \mu_N$: in this case g 's play role of variables of, say, iterated integrals (75), while t 's are formal parameters used to make the generating series. This symmetry is amplified by formulae for the coproduct given below: compare (82) and (83).

iii) Define

$$I_{n_1, \dots, n_m, n_{m+1}}(a_1 : \dots : a_m : a_{m+1}) \in \mathcal{D}_{w, m}(G) \quad \text{where} \quad w := n_1 + \dots + n_{m+1} - 1$$

as the coefficients of the generating series (54):

$$\{a_1 : \dots : a_m : a_{m+1} | t_1 : \dots : t_m : t_{m+1}\} := \quad (76)$$

$$\sum_{n_i > 0} I_{n_1, \dots, n_m, n_{m+1}}(a_1 : \dots : a_m : a_{m+1}) t_1^{n_1-1} \dots t_m^{n_m-1} t_{m+1}^{n_{m+1}-1}$$

More explicitly,

$$I_{n_1, \dots, n_m, n_{m+1}}(a_1 : \dots : a_{m+1}) = \quad (77)$$

$$(-1)^{n_{m+1}-1} \sum_{i_1 + \dots + i_m = n_{m+1}-1} \binom{n_1 + i_1}{i_1} \dots \binom{n_m + i_m}{i_m} I_{n_1+i_1, \dots, n_m+i_m}(a_1 : \dots : a_{m+1})$$

Here $i_k \geq 0$. We recover (53) for $n_{m+1} = 1$.

We obviously have the cyclic symmetry relations

$$I_{n_1, \dots, n_{m+1}}(a_1 : \dots : a_{m+1}) = I_{n_2, \dots, n_{m+1}, n_1}(a_2 : \dots : a_{m+1} : a_1) \quad (78)$$

When $G = \mu_N$ the properties of the generators $I_{n_1, \dots, n_m, n_{m+1}}(a_1 : \dots : a_{m+1})$ reflect the properties of the iterated integral

$$\int_0^{a_{m+1}} \underbrace{\frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_{m+1}-1} \circ \underbrace{\frac{dt}{a_1-t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_1} \circ \dots \circ \underbrace{\frac{dt}{a_m-t} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{n_m} \quad (79)$$

This integral is divergent if $n_{m+1} > 1$, and so has to be regularized (see theorem 7.1 in [G3] or [G1], or [G4]). Its regularized value is given by formula (77).

The interpretation of the $\{g : |t : \}$ -generating series on the circle is reminiscent of the structure of the iterated integral (79): the $\frac{dt}{t}$ differentials located between $\frac{dt}{t-a_i}$ and $\frac{dt}{t-a_{i+1}}$ correspond to the t_i -variable of the generating series, and thus sit between a_i and a_{i+1} on the circle.

4. The cobracket $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2 \mathcal{D}_{\bullet\bullet}(G)$. It will be defined by

$$\delta\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} = \quad (80)$$

$$- \sum_{k=2}^m \text{Cycle}_{m+1} \left(\{g_1 : \dots : g_{k-1} : g_k | t_1 : \dots : t_{k-1} : t_{m+1}\} \wedge \{g_k : \dots : g_{m+1} | t_k : \dots : t_{m+1}\} \right)$$

where the indices are modulo $m+1$ and

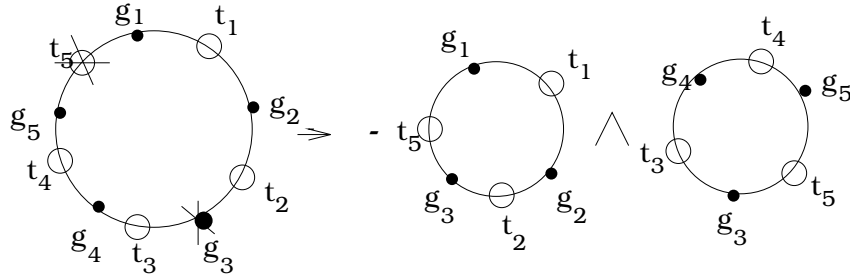
$$\text{Cycle}_{m+1} f(x_1, \dots, x_{m+1}) := \sum_{i=1}^{m+1} f(x_i, \dots, x_{m+i})$$

Remark. Let us add formally to the generators the coefficients of the generating series $\{g_1|t_1\}$ and put them equal to zero. Then the sum in (80) will be from 1 to $m+1$.

Each term of the formula corresponds to the following procedure: choose a slot and a dual slot on the circle. Cut the circle at the chosen slot and dual slot and make two oriented circles with a dihedral words on each of them out of the initial data. It is useful to think about the slots and dual slots as of little arcs, not points, so cutting one of them we get the arcs on each of the two new circles marked by the corresponding letters. The formula reads as follows:

$$\delta((80)) = - \sum_{\text{cuts}} (\text{start at the dual slot}) \wedge (\text{start at the slot})$$

The only asymmetry between g 's and t 's is the order of factors.



The cobracket on the generators looks as follows.

$$\delta I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1}) = \quad (81)$$

$$-\text{Cycle}_{m+1} \left(\sum_{k=2}^m \sum_{n'+n''=n_{m+1}+1} I_{n_1, \dots, n_{k-1}, n'}(g_1 : \dots : g_k) \wedge I_{n_k, \dots, n_m, n''}(g_k : \dots : g_{m+1}) \right)$$

Notice that the second summation is over positive integers n', n'' such that $(n' - 1) + (n'' - 1) = n_{m+1} - 1$.

Theorem 4.3 a) *There exists unique map $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2 \mathcal{D}_{\bullet\bullet}(G)$ for which (80) holds, providing a bigraded Lie coalgebra structure on $\mathcal{D}_{\bullet\bullet}(G)$.*

b) *A similar result is true for $\widehat{\mathcal{D}}_{\bullet\bullet}(G)$. Moreover there is an isomorphism of bigraded Lie algebras $\widehat{\mathcal{D}}_{\bullet\bullet}(G) = \mathcal{D}_{\bullet\bullet}(G) \oplus \mathbb{Q}(1, 1)$ where $\mathbb{Q}(1, 1)$ is a one dimensional Lie coalgebra of bidegree $(1, 1)$.*

Remark. The data of f in $V[[t_1, \dots, t_{m+1}]]$ is the same thing as the data, for any nilpotent ring A , and any nilpotent elements t_1, \dots, t_{m+1} in A of $f(t_1, \dots, t_{m+1}) \in A \otimes V$ functorial in A . Therefore to define $\delta : \mathcal{D}_{\bullet\bullet}(G) \longrightarrow \Lambda^2 \mathcal{D}_{\bullet\bullet}(G)$ it suffices

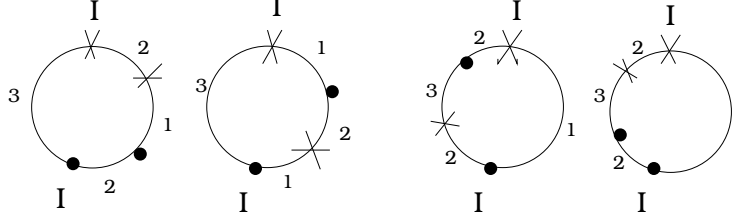
to show that the *function* of g, t with values in $\Lambda^2 \mathcal{D}_{\bullet\bullet}(G) \otimes A$ given by proposed $\delta\{g|t\}$ obeys the relations (i)-(v).

Proof. Let \mathcal{E} be the \mathbb{Q} -vector space generated by the coefficients of the $\{g : |t : \}$ -generating series submitted only to the cyclic invariance (64). To define a map $\mathcal{E} \longrightarrow V$ amounts to give generating series $\{g : |t : \}$ in $V[[t_1, \dots]]$ obeying (64), and the fact that (80) defines a map $\mathcal{E} \longrightarrow \Lambda^2 \mathcal{E}$ is clear.

The Jacobi identity $\delta\delta = 0$. It holds in \mathcal{E} . One has to prove that $\delta\delta\{g_1 : \dots : g_{m+1}|t_1 : \dots : t_{m+1}\} = 0$ in $\Lambda^3 \mathcal{E}[[t_1, \dots, t_{m+1}]]$.

Let $(1) \wedge (2)$ be a single term in (80) corresponding to cuts at given slot and dual slot on the circle. Let us show that $\delta(1) \wedge (2) - (1) \wedge \delta(2) = 0$: this implies the Jacobi identity. The terms in the expression $\delta(1) \wedge (2)$ could be of two different types, corresponding to the two situations shown on the left part of the picture. Here we marked by I the initial cuts, so for $\delta(1) \wedge (2)$ the new cuts must be after the marked by I dual slot. The four cuts define four arcs on the circle, and computing $\delta(1) \wedge (2)$ we make three little circles out of them, wedged in a certain order. The numbers 1, 2, 3 on the arcs indicate this order.

The terms in $(1) \wedge \delta(2)$ correspond to the two drawings on the right of the picture. It is easy to see that the terms $N1$ and $N4$, and the terms $N2$ and $N3$, cancel each other (provided the cuts on the corresponding circles are made in the same slots and dual slots).



It is easy to see that adding relations (61) we still get a Lie coalgebra.

The shuffle relations. We will show that the cyclic symmetry relations (64) together with each of the shuffle relations generate a coideal.

We start from the shuffle relations (63). Let us impose (56) and (63) to define the quotient \mathcal{E}_1 of \mathcal{E} . To define a map $\mathcal{E}_1 \longrightarrow V$ amounts to define $\{\dots\}$ in $V[[t_1, \dots]]$ obeying defining relations of \mathcal{E} plus (56) and (63).

We will need an explicit description of the cobracket for the generating series $\{g_1 : |t_1, \}$ and $\{g_1, |t_1 : \}$.

Lemma 4.4 *One has*

$$\begin{aligned} a) \quad & \delta\{g_1 : \dots : g_{m+1}|t_1, \dots, t_{m+1}\} := \\ & - \sum_{k=2}^m \text{Cycle}_{m+1} \left(\{g_1 : \dots : g_{k-1} : g_k|t_1, \dots, t_{k-1}, x_k\} \wedge \right. \end{aligned} \tag{82}$$

$$\begin{aligned}
& \left. \{g_k : g_{k+1} : \dots : g_{m+1} | y_k, t_{k+1}, \dots, t_{m+1}\} \right) \\
& \text{where } t_1 + \dots + t_{k-1} + x_k = 0, y_k + t_{k+1} + \dots + t_{m+1} = 0. \\
& b) \quad \delta\{g_1, \dots, g_{m+1} | t_1 : \dots : t_{m+1}\} = \\
& \sum_{k=2}^m \text{Cycle}_{m+1} \left(\{g_1, \dots, g_{k-1}, a_k | t_1 : \dots : t_{k-1} : t_k\} \wedge \right. \\
& \left. \{b_k, g_{k+1}, \dots, g_{m+1} | t_k : t_{k+1} : \dots : t_{m+1}\} \right) \\
& \text{where } a_k \text{ and } b_k \text{ satisfy the conditions } g_1 \dots g_{k-1} a_k = 1 \text{ and } b_k g_{k+1} \dots g_{m+1} = 1.
\end{aligned} \tag{83}$$

Notice the sign difference between the otherwise similar (82) and (83). It might be explained by the different role the slots and dual slots play in the definition of δ , as well as in the $\{a, |b:\}$ and $\{a : |b,\}$ -generating series.

Proof. The proofs of these formulas are more or less identical, so we will present only the proof of the first one. We have

$$\begin{aligned}
& \delta\{g_1 : \dots : g_{m+1} | t'_1, \dots, t'_{m+1}\} = \delta\{g_1 : \dots : g_{m+1} | t_1 : \dots : t_{m+1}\} = \\
& - \sum_{k=2}^m \text{Cycle}_{m+1} \left(\{g_1 : \dots : g_{k-1} : g_k | t_1 : \dots : t_{k-1} : t_{m+1}\} \wedge \right. \\
& \left. \{g_k : g_{k+1} : \dots : g_{m+1} | t_k : t_{k+1} : \dots : t_{m+1}\} \right) = \\
& - \sum_{k=2}^m \text{Cycle}_{m+1} \left(\{g_1 : \dots : g_{k-1} : g_k | t'_1, t'_2, \dots, t'_{k-1}, -t_{k-1} + t_{m+1}\} \wedge \right. \\
& \left. \{g_k : g_{k+1} : \dots : g_{m+1} | t_k - t_{m+1}, t'_{k+1}, \dots, t'_{m+1}\} \right)
\end{aligned} \tag{84}$$

This is just what we wanted. The lemma is proved.

Recall that $(g_i | t_i)$ sits at the slot i . A shuffle relation (63) is determined by choosing a slot, say $m+1$, and integer $1 \leq p \leq m-1$. Namely, we shuffle the slots numbered by

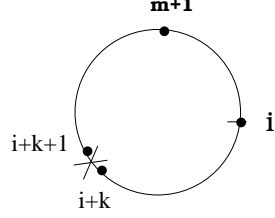
$$\{1, \dots, p\} \quad \text{and} \quad \{p+1, \dots, m\} \tag{85}$$

together with the symbols $(g_i | t_i)$ attached to them, and make generating series of type (56) out of each of the shuffles by putting it on the oriented circle before the slot $m+1$. We use a shorthand $s_1(1, \dots, p | p+1, \dots, m)$ for this shuffle relation.

Let us mark a slot i and the dual slot $i+k$, which is between the slots $i+k$ and $i+k+1$, see the picture. The marks determine a single term (the indices are modulo $m+1$)

$$(g_i : g_{i+1} : \dots : g_{i+k} | y_i, t_{i+1}, \dots, t_{i+k}) \wedge (g_{i+k+1} : \dots : g_{i-1} : g_i | t_{i+k+1}, \dots, t_{i-1}, x_i) \tag{86}$$

in formula (82) for the coproduct, denoted $\partial_{[i,k]}(\{g_0 : g_1 : \dots : g_m | t_0, t_1, \dots, t_m\})$. We will focus our attention on this term.



Assume first that $m+1 \notin \{i+1, \dots, i+k\}$. Consider all the shuffles σ of the sets (85) such that

$$\{\sigma(1), \dots, \sigma(p)\} \cap \{i+1, \dots, i+k\} \text{ is a given set } \{a, a+1, \dots, b\}$$

Then the complement to $\{a, a+1, \dots, b\}$ in $\{i, \dots, i+k\}$ is also a given set, denoted $\{\alpha, \dots, \beta\}$. We assume that the slots $\{a, \dots, b\}$ as well as $\{\alpha, \dots, \beta\}$ are located on the circle in the order prescribed by the orientation of the circle. Then the terms in $\partial_{[i,k]}(s_1(1, \dots, p|p+1, \dots, m))$ corresponding to such σ 's are in the subspace $s_1(a, \dots, b|\alpha, \dots, \beta) \wedge \{\text{all terms}\}$. If $m+1 \in \{i+1, \dots, i+k\}$ then working with the second term in (86) we may argue just as before.

Using formula (83) and formal symmetry $g < - > t$ we conclude that the cyclic symmetry together with shuffle relations (62) also generate a coideal.

The distribution relations. Assume first that there are no two consecutive equal elements among x_i . Then computing

$$\delta \frac{1}{|G_l|} \sum_{y_i^l = x_i} \{y_1 : \dots : y_{m+1} | t_1 : \dots : t_{m+1}\}$$

via formula (80), applying the distribution relations to the first factors, and after that to the second factors of $\delta(\dots)$ we get $\delta\{x_1 : \dots : x_{m+1} | t_1 : \dots : t_{m+1}\}$. However since the distribution relations for $\{e : e | t_1 : t_2\}$ hold only up to a constant we need additional arguments to show that the total contribution of these constants is zero.

Suppose $x_1 = x_2 = e$. Let us show that if $m \neq 2$ or $x_3 \neq e$ then

$$\{e : e | t_1 : t_{m+1}\} \wedge \{e : x_3 : \dots : x_{m+1} | t_2 : \dots : t_{m+1}\} + \quad (87)$$

$$\{x_3 : \dots : x_{m+1} : e | t_3 : \dots : t_{m+1} : t_2\} \wedge \{e : e | t_1 : t_2\} = \quad (88)$$

$$\frac{1}{(|G_l|)^2} \sum_{y_i^l = x_i} \left(\{y_1 : y_2' | t_1 : t_{m+1}\} \wedge \{y_2 : y_3 : \dots : y_{m+1} | t_2 : \dots : t_{m+1}\} + \quad (89)$$

$$\{y_3 : \dots : y_{m+1} : y_1' | t_3 : \dots : t_{m+1} : t_2\} \wedge \{y_1 : y_2 | t_1 : t_2\} \right) = \quad (90)$$

Here (87) + (88) is a sum of two appropriate terms in the formula for $\delta(\dots)$. We assume the summation over the arbitrary l -torsion elements y_1' and y_2' . Applying

the distribution relations we write the last two lines as

$$\begin{aligned}
& \frac{1}{|G_l|} \sum_{y_i^l = x_i} \left(\{e : e|t_1 : t_{m+1}\} \wedge \{y_2 : y_3 : \dots y_{m+1}|lt_2 : \dots : lt_{m+1}\} + \right. \\
& \quad \left. \{y_3 : \dots : y_{m+1} : y_1'|lt_3 : \dots : lt_{m+1} : lt_2\} \wedge \{e : e|t_1 : t_2\} \right) + \\
& + \frac{1}{|G_l|} \left(\left(\sum_{y_1^l = e} I_1(y_1 : e) - I_1(e : e) \right) \wedge \sum_{y_i^l = x_i} \{y_2 : y_3 : \dots y_{m+1}|lt_2 : \dots : lt_{m+1}\} + \right. \\
& \quad \left. \sum_{y_i^l = x_i} \{y_3 : \dots : y_{m+1} : y_1'|lt_3 : \dots : lt_{m+1} : lt_2\} \wedge \left(\sum_{y_1^l = e} I_1(y_1 : e) - I_1(e : e) \right) \right)
\end{aligned}$$

The last two lines cancel each other thanks to the skewsymmetry relation in Λ^2 , the cyclic symmetry

$$\{y_2 : y_3 : \dots y_{m+1}|lt_2 : \dots : lt_{m+1}\} = \{y_3 : \dots : y_{m+1} : y_2|lt_3 : \dots : lt_{m+1} : lt_2\}$$

and the observation that since $x_1 = x_2 = e$ we can replace summation over y_1^l by the summation over y_2 . If $m \neq 2$ or $x_3 \neq e$ we are done: applying the distribution relations to the first two lines we get (87) + (88). Suppose that $m = 2$ and $x_3 = e$. Then in the last step we get (87) + (88) plus the reminder terms

$$\begin{aligned}
& \{e : e|t_1 : t_3\} \wedge \left(\sum_{y_2^l = e} I_1(y_2 : e) - I_1(e : e) \right) + \\
& \left(\sum_{y_3^l = e} I_1(y_3 : e) - I_1(e : e) \right) \wedge \{e : e|t_1 : t_2\}
\end{aligned}$$

In this case to get the total contribution of the reminder terms one has to apply the cyclic summation Cycle_3 along the $(y_1|t_1), \dots, (y_3|t_3)$ variables. Therefore thanks to the skewsymmetry the total reminder term is zero. The distribution relations, and hence the part a) of the theorem are proved.

Proof of the part b). The coproduct on $\widehat{\mathcal{D}}_{\bullet\bullet}(G)$ is well defined thanks to the part a) of the theorem. The decomposition into a direct sum of bigraded vector spaces is true by the very definitions. It remains to check that

$$\delta(\widehat{\mathcal{D}}_{\bullet\bullet}(G)) \subset \Lambda^2 \mathcal{D}_{\bullet\bullet}(G)$$

i.e. $I_{1,1}(g : g)$ appears in $\delta I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1})$ with the factor equal to zero. To calculate this factor we need only the following component of $\delta I_{n_1, \dots, n_{m+1}}(g_1 : \dots : g_{m+1})$:

$$-\text{Cycle}_{m+1} \left(I_{n_1,1}(g_1 : g_2) \wedge I_{n_2, \dots, n_m, n_{m+1}}(g_2 : \dots : g_{m+1}) + \right.$$

$$I_{n_1, \dots, n_{m-1}, n_{m+1}}(g_1 : \dots : g_{m-1} : g_m) \wedge I_{n_m, 1}(g_m : g_{m+1})$$

We rewrite its second term as

$$-\text{Cycle}_{m+1} \left(I_{n_3, \dots, n_{m+1}, n_2}(g_3 : \dots : g_{m+1} : g_1) \wedge I_{n_1, 1}(g_1 : g_2) \right)$$

So if $g_1 = g_2$ the factor appearing at $I_{1,1}(g_1 : g_2)$ is zero thanks to the cyclic symmetry. The part b) is proved.

The dihedral Lie coalgebra $\mathcal{D}_\bullet(G|H)$. Let G and H be two (finite) commutative groups. We define a Lie coalgebra $\mathcal{D}_\bullet(G|H)$, graded by the integers $m \geq 1$, as follows. The \mathbb{Q} -vector space $\mathcal{D}_m(G|H)$ is generated by the symbols $\{g_1 : \dots : g_{m+1} | h_1 : \dots : h_{m+1}\}$ obeying the relations i), ii) and iii). We assume the homogeneity in both g 's and h 's. The distribution relations look as follows:

$$\sum_{s'_i = h_i} \{x_1 : \dots : x_m : 1 | s_1 : \dots : s_m : 1\} = \sum_{y'_i = x_i} \{y_1 : \dots : y_m : 1 | h_1 : \dots : h_m : 1\}$$

This definition has an obvious generalization when G or H are commutative group schemes. (The role of constant in t may play a constant function in $\mathbb{Q}[H]$).

Proof of the theorem 2.8. It is completely similar to the proof of theorem 4.3a): suppress t 's from the notations, and denote g 's by, say, v 's understood as appropriate vectors of the lattice L_m , and use the additive notations instead of the multiplicative. The the proof goes literally the same.

5. Two related bigraded Lie coalgebras: $\tilde{\mathcal{D}}_{\bullet\bullet}(G)$ and $\mathcal{D}'_{\bullet\bullet}(G)$.

i) The generators of $\tilde{\mathcal{D}}_{w,m}(G)$ are symbols

$$\tilde{I}_{n_0, \dots, n_m}(g_0 : \dots : g_m) \quad \sum n_i = w - 1, \quad n_i > 0, m \geq 1$$

We package them into the generating series

$$\{g_0 : \dots : g_m | t_0 : \dots : t_m\}^\sim := \sum_{n_i > 0} \tilde{I}_{n_0, \dots, n_m}(g_0 : \dots : g_m) t_0^{n_0} \dots t_m^{n_m} \quad (91)$$

The relations are the following: *the G -homogeneity*: for any $h \in G$

$$\{h g_0 : \dots : h g_m | t_0 : \dots : t_m\}^\sim = \{g_0 : \dots : g_m | t_0 : \dots : t_m\}^\sim \quad (92)$$

the cyclic symmetry (78), and in addition the constant term of (91) is zero when the g 's are all equal, i.e.

$$\tilde{I}_{1, \dots, 1}(e : \dots : e) = 0 \quad (93)$$

The first part of the proof of theorem 4.3 shows that formula (81) provides a bigraded Lie coalgebra structure on $\tilde{\mathcal{D}}_{\bullet\bullet}(G)$.

For A the associative algebra freely generated by a finite set S , the vector space $\mathcal{C}(A) := A/[A, A]$ has as basis the cyclic words in S . We denote by $\mathcal{C} : A \longrightarrow \mathcal{C}(A)$ the natural projection.

The algebra A is the universal enveloping algebra of the free Lie algebra generated by S , and as such has a commutative coproduct Δ . If we identify A with its graded dual, using the basis afforded by words in S , this coproduct dualizes into the shuffle product

$$(s_1 \dots s_k) \circ_{Sh} (s_{k+1} \dots s_{k+l}) := \sum_{\Sigma_{k,l}} s_{\sigma(1)} \dots s_{\sigma(k+l)} \quad (94)$$

Let $A(G)$ be the free associative algebra with the generators Y and X_g , $g \in G$. The group G acts on the generators of $A(G)$ as in s. 1.4. So it acts by automorphisms of $A(G)$. Let $\tilde{\mathcal{C}}(A(G))$ be the quotient of $\mathcal{C}(A(G))$ by the subspace generated by Y^n , X_g^n , $n \geq 0$.

Lemma 4.5 *There is a canonical isomorphism of \mathbb{Q} -vector spaces*

$$\eta : \tilde{\mathcal{D}}_{\bullet\bullet}(G) \longrightarrow \tilde{\mathcal{C}}(A(G))_G \quad (95)$$

$$\tilde{I}_{n_0, \dots, n_m}(g_0 : \dots : g_m) \longmapsto \mathcal{C}(X_{g_0} Y^{n_0-1} \dots X_{g_{m-1}} Y^{n_{m-1}-1} X_{g_m} Y^{n_m-1})$$

Proof. Cyclic invariance of \tilde{I} corresponds to cyclic words being considered, homogeneity in g for \tilde{I} corresponds to taking coinvariants and the relation (93) corresponds to the relation $X_g^n = 0$ ($n \geq 1$). We divided as well by Y^n ($n \geq 0$) which otherwise would have been left out of the image.

ii) Define a *shuffle relation* in $\tilde{\mathcal{D}}_{\bullet\bullet}(G)$ as the image under the isomorphism η^{-1} of

$$\mathcal{C}\left(X_e \cdot \left\{ (Y^{n_0-1} X_{g_1} Y^{n_1-1} \dots X_{g_k} Y^{n_k-1}) \circ_{Sh} (Y^{m_0-1} X_{h_1} Y^{m_1-1} \dots X_{h_l} Y^{m_l-1}) \right\}\right) \quad (96)$$

where the expressions in each of the parentheses (\cdot) are nonempty. We define $\mathcal{D}'_{\bullet\bullet}(G)$ as the quotient of $\tilde{\mathcal{D}}_{\bullet\bullet}(G)$ by the subspace generated by these shuffle relations.

iii) Let $\mathcal{D}''_{w,m}(G)$ be the \mathbb{Q} -vector space generated by the symbols (53) subject to the relations (61), (64), (63) and v). So it has the same generators as $\mathcal{D}_{w,m}(G)$, but the relations are relaxed.

Proposition 4.6 *There is an isomorphism $i : \mathcal{D}''_{w,m}(G) \longrightarrow \mathcal{D}'_{w,m}(G)$ given by*

$$i : I_{n_1, \dots, n_m}(g_1 : \dots : g_{m+1}) \longmapsto \tilde{I}_{n_1, \dots, n_m, 1}(g_1 : \dots : g_{m+1}) \quad (97)$$

Proof. *Comparison of the spaces \mathcal{D}' and \mathcal{D}'' .* Defining both in terms of the $\{g : |t : \}$ -generating series one requires:

for both: G -homogeneity, cyclic invariance, nullity of the constant terms of the $\{e : e : \dots : e\}$. (This holds for \mathcal{D}'' for $m = 1$ by the definition, and for $m > 1$ by the $\{g : |t, \}$ -shuffle relation);

for \mathcal{D}'' : t -homogeneity, $\{g : |t, \}$ -shuffle;

for \mathcal{D}' : shuffle (96).

The map i is well defined. Observe that (96) for $X_e \cdot \{Y \circ_{sh} (\dots)\}$ gives in \mathcal{D}' that

$$\partial_t \{g_0 : \dots : g_m | t_0 + t : \dots : t_m + t\} = 0 \quad \text{at } t = 0 \quad (98)$$

hence the t -homogeneity.

Finally, one can show that the $\{g : |t, \}$ -shuffle relations (63) go under the map i precisely to the space

$$\eta^{-1}(\text{the subspace of the shuffle relations (96) with } n_0 = m_0 = 1) \quad (99)$$

(In fact in [G3] relations (63) appeared as a way to write the shuffle relations (96) with $n_0 = m_0 = 1$).

Surjectivity of the map i . For $m = 0$ the t -homogeneity (98) implies $\{g|t\} = 0$, i.e. $\tilde{I}_n(g_0) = 0$ in \mathcal{D}' .

The t -homogeneity (98) in \mathcal{D}' allows to express $\tilde{I}_{n_0, n_1, \dots, n_m}(g_0 : g_1 : \dots : g_m)$ via $\tilde{I}_{1, n_1, \dots, n_m}(g_0 : g_1 : \dots : g_m)$ for $m \geq 1$. More precisely, let S be the subspace of $A(G)_G$ generated by elements $X_e \{Y \circ_{sh} \mathcal{A}\}$. We write $a \stackrel{S}{=} b$ if $a - b \in S$. Then t -homogeneity is equivalent to the formula

$$\begin{aligned} X_e Y^{n_0-1} X_{g_1} Y^{n_1-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1} &\stackrel{S}{=} \\ (-1)^{n_0-1} X_e X_{g_1} \{ (Y^{n_0-1}) \circ_{sh} (Y^{n_1-1} X_{g_2} Y^{n_2-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1}) \} \end{aligned} \quad (100)$$

So the map i is surjective.

Remark. Applying η^{-1} to formula (100) we get relations (77).

Injectivity of the map i . We need to show that general shuffle relations (96),

$$X_e \{ (Y^{n_0-1} X_{g_1} \cdot \dots) \circ_{sh} (Y^{m_0-1} X_{h_1} \cdot \dots) \},$$

belong to the subspace of $A(G)_G$ generated by S and the shuffle relations (96) with $n_0 = m_0 = 1$. Indeed, by (100)

$$X_e \{ Y^{n_0-1} X_{g_1} \cdot \dots \} = X_e \{ (Y \circ_{sh} \mathcal{A}) \} + X_e \{ X_{g_1} \cdot \dots \}$$

Since obviously $X_e \{ (Y \circ_{sh} \mathcal{A}_1) \circ_{sh} \mathcal{A}_2 \} = X_e \{ Y \circ_{sh} (\mathcal{A}_1 \circ_{sh} \mathcal{A}_2) \}$ we see that for $\mathcal{B} := Y^{m_0-1} X_{h_1} \cdot (\dots)$ one has

$$X_e \{ (Y^{n_0-1} X_{g_1} \cdot \dots) \circ_{sh} \mathcal{B} \} \stackrel{S}{=} X_e \{ (X_{g_1} \cdot \dots) \circ_{sh} \mathcal{B} \}$$

Similarly $X_e \{ \mathcal{B} \} \stackrel{S}{=} X_e \{ X_{h_1} \cdot \dots \}$, thus

$$X_e \{ \mathcal{B} \circ_{sh} (X_{g_1} \cdot \dots) \} \stackrel{S}{=} X_e \{ (X_{h_1} \cdot \dots) \circ_{sh} (X_{g_1} \cdot \dots) \}$$

The statement, and hence the part b) of the theorem are proved.

Proposition 4.7 *The quotient $\mathcal{D}'_{\bullet\bullet}(G)$ of $\widetilde{\mathcal{D}}_{\bullet\bullet}(G)$ inherits from $\widetilde{\mathcal{D}}$ a cobracket.*

Proof. Formulas (80) and (81) define a cobracket on \mathcal{D}'' . This cobracket, transported by i to \mathcal{D}' , is induced from $\widetilde{\mathcal{D}}$. For another proof see in the end of s. '6.

The Lie coalgebra $\mathcal{D}_{\bullet\bullet}(G)$ is a quotient of $\mathcal{D}'_{\bullet\bullet}(G)$.

5 The dihedral Lie algebras and special equivariant derivations

1. The special derivations and cyclic words (after [Dr], [K]). Let A be the free associative algebra generated by a finite set S . We are going to define a Lie algebra $\text{Der}^S A$ of special derivations of A and describe it via cyclic words in S . A derivation D of the algebra A is called special if there are elements $B_s \in A$ such that

$$D(s) = [B_s, s] \quad \text{and} \quad D\left(\sum_{s \in S} s\right) = 0 \quad (101)$$

Thus

$$\text{a system } B = \{B_s\} \text{ of elements of } A \text{ with } \sum [B_s, s] = 0 \quad (102)$$

defines a special derivation D_B . One has

$$[D_B, D_C] = D_{[B, C]}, \quad \text{where} \quad [B, C]_s := D_B(C_s) - D_C(B_s) - [B_s, C_s] \quad (103)$$

Formula (103) defines a Lie bracket on systems (B_s) obeying (102). Indeed, as was pointed out to me by the referee, one can interpret them as infinitesimal automorphisms of the structure consisting of

algebra A , for each $s \in S$, left module A (noted $A(s)$),

endomorphism $a \mapsto as$ of $A(s)$, element $\sum s$ in A

The bracket (103) is the bracket coming from this interpretation.

Define a map of \mathbb{Q} -linear spaces

$$\text{Cycl} : \mathcal{C}(A) \longrightarrow A : \quad s_1 \dots s_k \longmapsto \sum_{i=1}^k s_i \dots s_{i-1+k}, \quad 1 \longmapsto 0$$

given on the generators by the sum of the cyclic permutations. Then set

$$\partial_s : s_1 \dots s_k \longmapsto \begin{cases} s_2 \dots s_k & \text{if } s = s_1 \\ 0 & \text{otherwise} \end{cases}$$

Similarly define the map ∂'_s by $s_1 \dots s_k \mapsto s_1 \dots s_{k-1}$ if $s_k = s$ and 0 otherwise. Then the image of the map Cycl is precisely $\cap \text{Ker}(\partial_s - \partial'_s)$ in the positive degree part A^+ of A . If x is in this image

$$\sum [\partial_s x, s] = \sum \partial_s x \cdot s - \sum s \cdot \partial_s x = \sum \partial'_s x \cdot s - \sum s \cdot \partial_s x = x - x = 0 \quad (104)$$

Set $\mathcal{D}_s := \partial_s \circ \text{Cycl}$. Let us define a Lie bracket in $\mathcal{C}(A)$ by

$$[C_1, C_2] := - \sum_{s \in S} \mathcal{C} \left([\mathcal{D}_s C_1, \mathcal{D}_s C_2] \cdot s \right) \quad (105)$$

Then $\mathcal{C}(A)$ is the product of the central $\mathbb{Q} \cdot 1$ and of $\mathcal{C}^+(A)$. The map

$$\kappa' : \mathcal{C}^+(A) \longrightarrow \text{the Lie algebra (102) (103);} \quad \kappa' : x \mapsto \mathcal{D}_s(x)$$

is an isomorphism of \mathbb{Q} -vector spaces. This follows from theorem 4.2 in [K]. For the convenience of the reader we reproduce the argument. Let $F^1(A) := \oplus_{s \in S} A \otimes ds$. There is a sequence

$$0 \longrightarrow \mathcal{C}^+(A) = A/([A, A] + \mathbb{Q} \cdot 1) \xrightarrow{d} F^1(A) \xrightarrow{t} [A, A] \longrightarrow 0 \quad (106)$$

where $d(\mathcal{C}) := \sum \mathcal{D}_s \mathcal{C} \otimes ds$ and $t[a \otimes ds] := [a, s]$. It is a complex by (104), and it is clearly exact from the left and right. There is an isomorphism of graded (by the weight) \mathbb{Q} -vector spaces $A/\mathbb{Q} \cdot 1 \longrightarrow F^1(A)$, $s_1 \dots s_m \longrightarrow s_1 \dots s_{m-1} \otimes ds_m$. Thus the Euler characteristic of the weight $\geq -m$ part of (106) is zero. So the complex is exact.

The map κ' respects the Lie brackets. In particular this proves that (105) satisfies the Jacobi identity. Thus κ' is a Lie algebra isomorphism.

The centralizer of s in A is $\mathbb{Q}[s]$. So $\text{Ker} \kappa' = \oplus_{s \in S} \mathbb{Q}[s]$. Let

$$\tilde{\mathcal{C}}(A) := \mathcal{C}(A) / \oplus_{s \in S} \mathbb{Q}[s] \quad (107)$$

The map κ' induces a Lie algebra morphism $\kappa : \tilde{\mathcal{C}}(A) \longrightarrow \text{Der}^S(A)$.

Proposition 5.1 *The morphism $\kappa : \tilde{\mathcal{C}}(A) \longrightarrow \text{Der}^S(A)$ is an isomorphism.*

Proof. We need only to show that κ is surjective. This follows from exactness of (106).

2. The Lie algebra of special equivariant derivations. Now assume $S = \{0\} \cup G$ where G is a finite commutative group. The corresponding algebra A is the algebra $A(G)$ of 2.4, with generators Y and X_g ($g \in G$). Then $A(G)$ is graded by the weight and depth, and as before $\text{Der}^S A(G)$ inherits a weight grading compatible with a depth filtration. On $\tilde{\mathcal{C}}(A(G))$ it corresponds to the weight grading and depth filtration induced from $A(G)$, both shifted by one. It is clear for the weight grading. For depth: if x in $\tilde{\mathcal{C}}(A)$ has depth m , $y := \text{Cycl}(x)$

(in $A/\oplus_s \mathbb{Q}[s]$) has the same depth. If no $\partial_g y$ has depth $m-1$, it follows that y has no word of depth m except possibly X_0^k in the case $m=0$, which is zero in $\tilde{\mathcal{C}}(A(G))$ thanks to (107).

The map κ provides a linear map

$$\kappa_G : \tilde{\mathcal{C}}(A(G))^G \longrightarrow \text{Der}^S A(G)^G =: \text{Der}^{SE} A(G)$$

Proposition 5.1 implies that it is an isomorphism.

We will identify the vector space $\tilde{\mathcal{C}}(A(G))^G$ with its graded for the depth filtration: the graded for the depth filtration of the Lie algebra $\tilde{\mathcal{C}}(A(G))^G$ is $\tilde{\mathcal{C}}(A(G))^G$ with the Lie bracket given by the sum (105), with $s=0$ omitted:

$$[C_1, C_2] := - \sum_{g \in G} \mathcal{C}([D_g C_1, D_g C_2] \cdot X_g)$$

3. Formulation of the result. Recall that Y and X_g , $g \in G$ are the generators of the algebra $A(G)$. Let

$$\mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})^G := \sum_{h \in G} \mathcal{C}(X_{hg_0} Y^{n_0-1} \cdot \dots \cdot X_{hg_m} Y^{n_m-1})$$

The expressions on the left are parametrized by *cyclic G -equivariant words*, i.e. a G -orbits on the set of all cyclic words. Consider the following formal expression:

$$\hat{\xi}_G := \sum \frac{1}{|\text{Aut}\mathcal{C}|} \tilde{I}_{n_0, \dots, n_m}(g_0 : \dots : g_m) \otimes \mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})^G \quad (108)$$

where the sum is over all cyclic G -equivariant words \mathcal{C} in X_g, Y . The weight $1/|\text{Aut}\mathcal{C}|$ of a given cyclic word \mathcal{C} is the order of automorphism group of this cyclic word, as was pointed out by the referee.

Applying the map $Id \otimes \text{Gr}(\kappa)$ we get a bidegree $(0,0)$ element

$$\tilde{\xi}_G \in \tilde{D}_{\bullet\bullet}(G) \hat{\otimes}_{\mathbb{Q}} \text{GrDer}^{SE}_{\bullet\bullet} A(G)$$

Since G is finite $\mathcal{D}_{w,m}(G)$ is finite dimensional \mathbb{Q} -vector space. Let $D_{-w,-m}(G) = \mathcal{D}_{w,m}(G)^\vee$. Then $D_{\bullet\bullet}(G) := \oplus_{w,m \geq 1} D_{-w,-m}(G)$ is a bigraded Lie algebra. We similarly define $\tilde{D}_{\bullet\bullet}(G)$ and $D'_{\bullet\bullet}(G)$.

We may view the element $\tilde{\xi}_G$ as a map between the bigraded \mathbb{Q} -vector spaces:

$$\tilde{\xi}_G \in \text{Hom}_{\mathbb{Q}\text{-Vect}}(\tilde{D}_{\bullet\bullet}(G), \text{GrDer}^{SE}_{\bullet\bullet} A(G)) \quad (109)$$

Notice that $D_{\bullet\bullet}(G) \subset D'_{\bullet\bullet}(G) \subset \tilde{D}_{\bullet\bullet}(G)$.

Theorem 5.2 a) $\tilde{\xi}_G : \tilde{D}_{\bullet\bullet}(G) \xrightarrow{\cong} \text{GrDer}^{SE}_{\bullet\bullet} A(G)$ is an isomorphism of bigraded Lie algebras.

b) Restricting $\tilde{\xi}_G$ to $D'_{\bullet\bullet}(G)$ we get an isomorphism

$$\xi'_G : D'_{\bullet\bullet}(G) \xrightarrow{=} \text{GrDer}_{\bullet\bullet}^{SE} L(G) \quad (110)$$

c) Restricting ξ'_G to $D_{\bullet\bullet}(G)$ we get an injective Lie algebra morphism

$$\xi_G : D_{\bullet\bullet}(G) \hookrightarrow \text{GrDer}_{\bullet\bullet}^{SE} L(G) \quad (111)$$

4. Proof. a) We start with a remark from linear algebra. Let L_1, L_2 be Lie algebras and $\zeta \in \text{Hom}_{\mathbb{Q}\text{-Vect}}(L_1, L_2) = L_1^* \otimes L_2$. Denote by $\delta : L_1^* \rightarrow \Lambda^2 L_1^*$ the Lie cobracket on L_1^* and by $[\cdot, \cdot] : \Lambda^2 L_2 \rightarrow L_2$ the Lie bracket on L_2 . Define a symmetric product $(L_1^* \otimes L_2) \otimes (L_1^* \otimes L_2) \rightarrow \Lambda^2 L_1^* \otimes \Lambda^2 L_2$ by $(a_1 \otimes a_2) \circ (b_1 \otimes b_2) := (a_1 \wedge b_1) \otimes (a_2 \wedge b_2)$. Then ζ is a morphism of Lie algebras if and only if

$$(\delta \otimes id)(\zeta) = \frac{1}{2}(id \otimes [\cdot, \cdot])(\zeta \circ \zeta)$$

If $\zeta = \sum_i A_i \otimes B_i$ then $(id \otimes [\cdot, \cdot])(\zeta \circ \zeta) = \sum_{i,j} A_i \wedge A_j \otimes [B_i, B_j]$ and so the condition is

$$\sum_i \delta(A_i) \otimes B_i = \frac{1}{2} \sum_{i,j} A_i \wedge A_j \otimes [B_i, B_j] \quad (112)$$

To check that $1/2$ is needed notice that if $\{e_i\}$ be a basis of a Lie algebra L , $\{f^i\}$ the dual basis, and $[e_i, e_j] = \sum c_{ij}^k e_k$, then $\delta f^k = \frac{1}{2} \sum_{i,j=1}^n c_{ij}^k f^i \wedge f^j$.

Let us suppose in addition that the Lie algebra bracket $[\cdot, \cdot]$ on L_2 is obtained by alternation of a (non necessarily associative) product $*$ on the vector space L_2 , i.e $[x, y] := x * y - y * x$. Then ζ is a Lie algebra morphism if and only if

$$(\delta \otimes id)(\zeta) = (id \otimes *) (\zeta \circ \zeta)$$

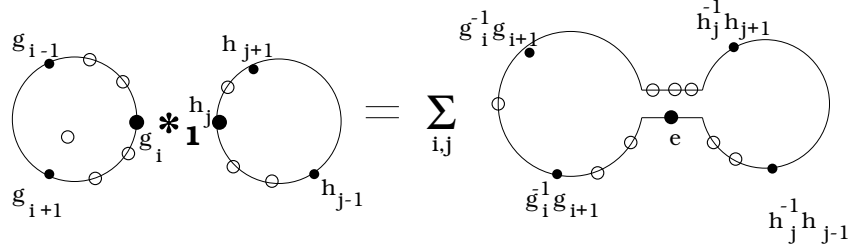
If $\zeta = \sum_i A_i \otimes B_i$ then it looks as follows

$$\sum_i \delta(A_i) \otimes B_i = \sum_{i,j} A_i \wedge A_j \otimes B_i * B_j \quad (113)$$

Let us return to our situation. The Lie bracket on $\text{Gr}\tilde{\mathcal{C}}(A(G))$ is given by formula (105) where the sum is over X_g only. It follows that the Lie algebra structure on the subspace $\text{Gr}\tilde{\mathcal{C}}(A(G))^G$ is obtained by alternation of the non associative product $*_1$ given by (see the picture below)

$$\begin{aligned} & \mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_k} Y^{n_k-1})^G *_1 \mathcal{C}(X_{h_0} Y^{m_0-1} \cdot \dots \cdot X_{h_l} Y^{m_l-1})^G := \\ & \sum_{i,j} \mathcal{C}(X_e Y^{n_i-1} X_{g_i^{-1}g_{i+1}} Y^{n_{i+1}-1} \cdot \dots \cdot X_{g_i^{-1}g_{i-1}} Y^{n_i+m_j-2} \cdot \\ & \quad X_{h_j^{-1}h_{j+1}} Y^{m_{j+1}-1} \cdot \dots \cdot X_{h_j^{-1}h_{j-1}} Y^{m_{j-1}-1})^G \end{aligned}$$

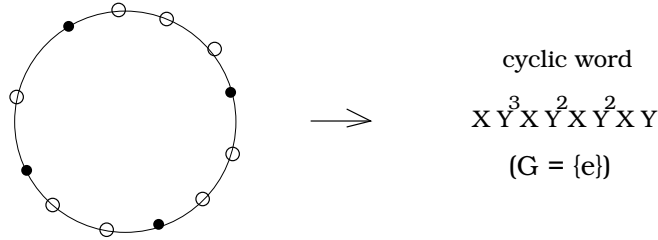
It can be defined by the same formulae for any abelian group G .



To visualize a single expression

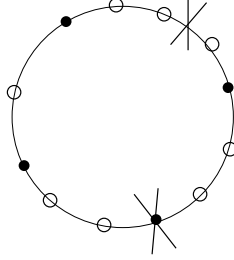
$$\delta(\tilde{I}_{n_0, n_1, \dots, n_m})(g_0 : \dots : g_m) \otimes \mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})^G \quad (114)$$

in $(\delta \otimes id)(\hat{\xi}_G)$ we proceed as follows. Take an oriented circle divided into $m+1$ arcs by $m+1$ black points. Label each of the black points by an element of the set $\{X_g\}$, $g \in G$ and call it an X_g -point. The i -th arc is subdivided into n_i little arcs by $n_i - 1$ points labeled by Y (presented by little circles on the picture and called Y -points). Such a data corresponds to a cyclic word $\mathcal{C}(X_{g_0} Y^{n_0-1} \cdot \dots \cdot X_{g_m} Y^{n_m-1})$.



The group G acts naturally on them: an element $h \in G$ transforms X_g -points to X_{hg} -points and leaves untouched Y -points. The orbits are called *circles with (n_0, \dots, n_m) -cyclic G -structure*. They correspond to cyclic G -equivariant words, and thus to expressions (114), as well as to the generators $\tilde{I}_{n_0, n_1, \dots, n_m}(g_0 : \dots : g_m)$ obeying the cyclic symmetry and homogeneity condition (92).

Let us mark such a picture by choosing a little arc and a black point different from the ends of the arc containing the little arc. We call it a *marked circle with (n_0, \dots, n_m) -cyclic G -structure*. They correspond to *marked cyclic G -equivariant words*.



a marked circle with
(3,2,2,1)-cyclic structure

It follows from formula (81) that expression (114) is a sum of the terms which are in bijective correspondence with markings of the particular circle which corresponds to $\mathcal{C}(X_{g_0}Y^{n_0-1}\dots)^G$. For example the marked circle with (3, 2, 2, 1)-cyclic structure on the picture corresponds to $I_{2,1} \wedge I_{2,1,2} \otimes \mathcal{C}(XY^2XYXY^3XY^2)$. In general we use the marks to cut the circle on 2 oriented semicircles and make 2 new circles by gluing the endpoints of the semicircles, adding a new black point on each of the new circles instead of the marked black point on the initial circle, and using the rest of the points on each of the new circles. The new circles are getting natural cyclic structures.

Going from a single expression (114) to the sum, weighted by $1/\text{Aut}\mathcal{C}$, of such expressions over all isomorphism classes of *cyclic G -equivariant words* we get the sum over all *marked cyclic G -equivariant words*. Notice that marking a particular cyclic word \mathcal{C} we get a sum of marked cyclic words, each of them appearing precisely $|\text{Aut}\mathcal{C}|$ times. The weight makes the coefficient equal to 1.

Now let us investigate how the right hand side of (113) looks for our element $\hat{\zeta}_G$. Consider the sum

$$\sum \frac{1}{|\text{Aut}\mathcal{C}_P|} \frac{1}{|\text{Aut}\mathcal{C}_Q|} \tilde{I}_{p_0, \dots, p_k}(g_0 : \dots : g_k) \wedge \tilde{I}_{q_0, \dots, q_l}(h_0 : \dots : h_l) \otimes \quad (115)$$

$$\mathcal{C}(X_{g_0}Y^{p_0-1} \cdot \dots \cdot X_{g_k}Y^{p_k-1})^G *_1 \mathcal{C}(X_{h_0}Y^{q_0-1} \cdot \dots \cdot X_{h_l}Y^{q_l-1})^G \quad (116)$$

where the summation is over all (ordered) pairs of cyclic G -equivariant words $\mathcal{C}_P = \mathcal{C}(X_{g_0}Y^{p_0-1} \cdot \dots \cdot X_{g_k}Y^{p_k-1})^G$ and $\mathcal{C}_Q = \mathcal{C}(X_{h_0}Y^{q_0-1} \cdot \dots \cdot X_{h_l}Y^{q_l-1})^G$.

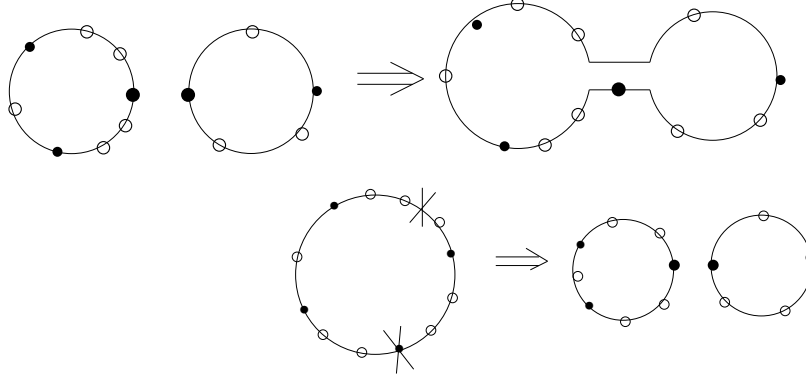
The particular product (116) is a sum of $(k+1)(l+1)$ cyclic G -equivariant words. Each of them corresponds to a pair

$$\{ \text{a circle with } P := (p_0, \dots, p_k)\text{-cyclic } G\text{-structure} + \text{a black point on it}, \\ \text{a circle with } Q := (q_0, \dots, q_l)\text{-cyclic } G\text{-structure} + \text{a black point on it} \}$$

We call a circle with cyclic G -structure and a choice of a black point on it a *labelled circle with cyclic G -structure*. It corresponds to a *labelled cyclic G -equivariant word*.

Now going from a single expression (116) to the weighted sum over all pairs of cyclic G -equivariant words we get the terms corresponding to pairs of labelled cyclic G -equivariant words, each with the coefficient 1 thanks to the weighting.

The pairs of *labelled* cyclic G -equivariant words are in bijective correspondence with the *marked* cyclic G -equivariant words, as demonstrated on the picture using labelled or marked circles instead of the corresponding words:



Namely, to get a marked circle with cyclic equivariant G -structure we proceed as follows. Make a connected sum of the P and Q -circles by connecting the chosen black points by a bridge. The orientation of the initial circles induces an orientation of their connected sum. Instead of the chosen two black points on the initial circles we put a single black point on the bottom part of the bridge. We keep the rest of the points, getting a cyclic structure on the new circle. The black point on the bottom bridge together with the top part of the bridge provide the marks on the circle with the cyclic structure we get. The distinguished point on each of the three circles is chosen to be the X_e -point. The procedure is reversible; the inverse map is shown on the bottom of the picture.

We proved that $\tilde{\xi}_G$ is a morphism of bigraded Lie algebras. By lemma 4.5 it is an isomorphism. The part a) of the theorem is proved.

Proof of the part b).

Lemma 5.3 *Let $D \in \text{Der}^{SE} A(G)$. Then $D(X_e) \in L(G) \iff D \in \text{Der}^{SE} L(G)$.*

Proof. We need the following fact. Let the free associative algebra $A(S)$ contain the free Lie algebra $L(S)$. Then, if $x \in A(S)$ is such that $[x, s] \in L(S)$, one has $x \in L(S) + \mathbb{Q}[s]$. Indeed, by Poincaré-Birkhoff-Witt, it suffices to check that the kernel of ads acting on $\text{Sym}^n(L(S))$ is reduced to s^n . Indeed, as $\mathcal{U}(\mathbb{Q} \cdot s)$ module, $L(S)$ is the sum of $\mathbb{Q}s$ and of a free module, and $L(S)^{\otimes n}$ hence the sum of $s^{\otimes n}$ and a free module.

We are now ready to prove the only nontrivial implication, \implies , of the lemma. Since D is equivariant $D(X_e) \in L(G)$ implies that $D(X_g) \in L(G)$ for any $g \in G$. Since $D(\sum X_g + Y) = 0$ we get also $D(Y) \in L(G)$. Thus thanks to the statement we just proved $D \in \text{Der}^{SE} L(G)$. The lemma is proved.

Recall that $\tilde{\mathcal{C}}(G)$ is a bigraded vector space and there is the isomorphism

$$\kappa : \tilde{\mathcal{C}}(G) \longrightarrow \text{Der}^S A(G) \quad (117)$$

The *vector space* $\text{Der}^S A(G)$ admits, besides the weight grading, a depth grading, with D of degree d if all $D(X_g)$ are of homogeneous degree $d + 1$. Indeed, if $[B_0, Y] + \sum [B_g, X_g] = 0$, the same holds with the B_g (resp. B_0) replaced by their part of degree d (resp. $d + 1$). The isomorphism (117) is compatible with the weight and depth grading (after a shift by one in $\tilde{\mathcal{C}}(G)$). Further, $\text{Der}^S L(G) \hookrightarrow \text{Der}^S A(G)$, defined by “the $D(X_g)$ are primitive” is a bigraded subspace. The action of the group G preserves the gradings. Thus $\text{Der}^{SE} L(G) \hookrightarrow \text{Der}^{SE} A(G)$ is also a bigraded subspace. Notice that the Lie algebra structure on either sides of (117) does not respect the depth grading, only the depth filtration.

Let $X \in \tilde{D}_{-w, -m}(G)$. Then we claim that

$$\hat{\xi}_G(X) \in \text{Der}^{SE} L(G) \quad \Leftrightarrow \quad X \in \hat{D}'_{-w, -m}(G) \quad (118)$$

Recall that there is a coproduct Δ on $A(G)$ which dualizes to the shuffle product \circ_{sh} , see (94). Let $\overline{\Delta}(Z) := \Delta(Z) - 1 \otimes Z - Z \otimes 1$. Then $L(G) = \text{Ker} \overline{\Delta}$. To prove (118) consider the expression

$$(id \otimes \overline{\Delta} \circ \mathcal{D}_{X_e})(\hat{\xi}_G) \in \tilde{D}_{-w, -m}(G) \otimes (A(G) \otimes A(G)) \quad (119)$$

Choose two elements A, B of the natural basis in $A(G)$. Then $A \otimes B \in A(G) \otimes A(G)$ appears in $(id \otimes \overline{\Delta} \circ \mathcal{D}_{X_e})(\hat{\xi}_G)$ with coefficient $X_e(A \circ_{sh} B)$. This implies (118). So map (117) leads to an isomorphism of the bigraded *vector spaces*

$$\hat{\xi}_G : D'_{\bullet\bullet}(G) \longrightarrow \text{Der}^{SE} L(G)$$

After taking the associated graded for the depth filtration it becomes an isomorphism of bigraded *Lie algebras* denoted ξ'_G .

Summarizing we see that shuffle relations (96) are equivalent to the condition $\tilde{\xi}_G(D'_{\bullet\bullet}(G)) \subset \text{GrDer}^{SE}_{\bullet\bullet} L(G)$. The part b) of the theorem is proved.

Another proof of proposition 4.7. We just proved that ξ'_G is an isomorphism of bigraded \mathbb{Q} -vector spaces. Using a) (and an obvious fact that $\text{Der}^{SE} L(G)$ is a Lie algebra!) we conclude that $D'_{\bullet\bullet}(G)$ is a Lie subalgebra of $\tilde{D}_{\bullet\bullet}(G)$.

c) It follows from a), b) and propositions 4.6 and 4.7. The theorem is proved.

5. The Lie algebra of outer semi-special derivations. Let

$$\mathcal{X}_N := \{0\} \cup \{\infty\} \cup \mu_N$$

For every point $x \in \mathcal{X}_N$ there is a well defined conjugacy class in $\pi_1(X_N, z)$ provided by a little loop around x .

For any point $y \in \mathcal{X}_N$ there is a tangent vector v_y at y provided by the canonical coordinate t on X_N if $y \neq \infty$ and t^{-1} at ∞ . Similarly for the pro- l group $\pi_1^{(l)}(X_N, v_y)$ there is a well defined up to a conjugation map:

$$i_x : \mathbb{Z}_l(1) \longrightarrow \pi_1^{(l)}(X_N, v_y) \quad (120)$$

Therefore the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N}))$ acts on $\pi_1^{(l)}(X_N, v_y)$ preserving the the conjugacy classes of the maps i_x , $x \in \mathcal{X}_N$. More precisely, define a *semi-special automorphism* of the group $\pi_1^{(l)}(X_N, v_y)$ as the automorphism preserving all the conjugacy classes of the maps i_x for $x \in \mathcal{X}_N$. Denote by $\text{Aut}^{SS} \pi_1^{(l)}(X_N, v_y)$ the group of all semi-special automorphisms of $\pi_1^{(l)}(X_N, v_y)$. We define the group of *outer* semi-special automorphisms $\text{Out}^{SS} \pi_1^{(l)}(X_N)$ as the quotient of the group $\text{Aut}^{SS} \pi_1^{(l)}(X_N, v_y)$ modulo the inner automorphisms. The corresponding groups defined for different base points v_y are canonically isomorphic, so we can drop the base vector from the notations. We get a canonical homomorphism

$$\Psi_N : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) \longrightarrow \text{Out}^{SS} \pi_1^{(l)}(X_N) \quad (121)$$

Similarly we define the Lie algebra $\text{ODer}^{SS} \mathbb{L}(X_N)$ of outer semi-special derivations of the fundamental Lie algebra of X_N :

$$\text{ODer}^{SS} \mathbb{L}(X_N) := \frac{\text{Der}^{SS} \mathbb{L}(X_N, v_y)}{\text{InDer} \mathbb{L}(X_N, v_y)} \quad (122)$$

Here the denominator is the Lie algebra of all inner derivations, and the numerator is the Lie algebra of semi-special derivations, i.e. derivations preserving the conjugacy classes of loops around $x \in \mathcal{X}_N$.

For any point $x_0 \in \mathcal{X}_N$ there is a Lie subalgebra

$$\text{Der}^S \mathbb{L}(X_N, v_{x_0}) \hookrightarrow \text{Der}^{SS} \mathbb{L}(X_N, v_{x_0}) \quad (123)$$

of all derivations *special* with respect to the point x_0 . It consists of the semi-special derivations killing $i_{x_0}(\mathbb{Q}(1))$. Thus there is a natural Lie algebra homomorphism

$$p_{x_0} : \text{Der}^S \mathbb{L}(X_N, v_{x_0}) \longrightarrow \text{ODer}^{SS} \mathbb{L}(X_N)$$

It is obviously surjective. Indeed, if \mathcal{D} is a semi-special derivation then there exists $C \in \mathbb{L}(X_N, v_{x_0})$ such that $\mathcal{D}(i_{x_0}(\mathbb{Q}(1))) = [C, i_{x_0}(\mathbb{Q}(1))]$, so subtracting the inner derivation $X \mapsto [C, X]$ we get a derivation special with respect to x_0 . Since the pro-nilpotent Lie algebra $\mathbb{L}(X_N, v_{x_0})$ is free, $\text{Ker} p_{x_0}$ is one dimensional, and consists of inner derivations provided by commutator with $i_{x_0}(\mathbb{Q}(1))$.

There is also the \mathbb{Q}_l -version of this story. We denote by $\text{ODer}^{SS} \mathbb{L}^{(l)}(X_N)$ the l -adic analog of (122).

6. A symmetric construction of the Lie algebra of special derivations. Below we present a variation on the theme discussed in the previous subsection, in a bit different (and general) setup.

Denote by $A_{\mathcal{X}}$ the free associative algebra generated by the set \mathcal{X} . Denote by X_x the generator corresponding to $x \in \mathcal{X}$. Let \mathcal{X} be a finite set. Choose an element $x_0 \in \mathcal{X}$. Let $S := \mathcal{X} - \{x_0\}$. Then there is the Lie algebra $\text{Der}^S(A_S)$ of special derivations of the algebra A_S . It contains the one dimensional subspace generated by inner derivation $* \mapsto [X_{x_0}, *]$. Our goal is to describe the structure of the quotient of the Lie algebra $\text{Der}^S(A_S)$ by this subspace in terms of the set \mathcal{X} independently of the choice of an element x_0 . We will use this for $\mathcal{X} := \mathcal{X}_N$ to describe the corresponding quotient of the Lie algebras $\text{Der}^{S\mathbb{L}}(X_N, v_y)$, where $y \in \mathcal{X}_N$. As a result we get natural *canonical isomorphisms* between these quotients.

Let $\overline{A}_{\mathcal{X}}$ be the quotient of $A_{\mathcal{X}}$ by the ideal $I(\mathcal{X})$ generated by the element $\sum_{x \in \mathcal{X}} X_x$. Then for any element $x \in \mathcal{X}$ there is a canonical isomorphism

$$\overline{A}_{\mathcal{X}} \xrightarrow{\cong} A_S \quad (124)$$

A semi-special derivation of the algebra $\overline{A}_{\mathcal{X}}$ is a derivation \mathcal{D} preserving the conjugacy classes of the generators X_x , i.e. $\mathcal{D}(X_x) = [C_x, X_x]$ for certain $C_x \in \overline{A}_{\mathcal{X}}$. The semi-special derivations form a Lie algebra $\text{Der}^{SS}(A_{\mathcal{X}})$. Let $\text{InDer}(\overline{A}_{\mathcal{X}})$ be the Lie subalgebra of inner derivations. The quotient

$$\text{ODer}^{SS}(\overline{A}_{\mathcal{X}}) := \frac{\text{Der}^{SS}(\overline{A}_{\mathcal{X}})}{\text{InDer}(\overline{A}_{\mathcal{X}})}$$

is called the Lie algebra of outer semi-special derivations. There is canonical surjective homomorphism

$$p_{x_0} : \text{Der}^S A_S \longrightarrow \text{ODer}^{SS}(\overline{A}_{\mathcal{X}})$$

whose kernel is the subspace generated by the inner derivations corresponding to elements $(\sum_{x \in \mathcal{X}} X_x)^n$, $n \geq 1$.

There is a similar story for the Lie algebras. Namely, let $L_{\mathcal{X}}$ be the free Lie algebra generated by the set \mathcal{X} . Denote by $\overline{L}_{\mathcal{X}}$ its quotient by the ideal generated by the element $\sum_{x \in \mathcal{X}} X_x$. Then for any $x \in \mathcal{X}$ there is canonical isomorphism $\overline{L}_{\mathcal{X}} \longrightarrow L_S$. We define the Lie algebra of outer semi-special derivations $\text{ODer}^{SS}(\overline{L}_{\mathcal{X}})$. There is a canonical surjective morphism

$$p_{x_0} : \text{Der}^S L_S \longrightarrow \text{ODer}^{SS}(\overline{L}_{\mathcal{X}})$$

with one dimensional kernel generated by the inner derivation given by commutator with $\sum_{x \in \mathcal{X}} X_x$.

Denote by $\widetilde{\mathcal{C}}(A_S)$ the quotient of the Lie algebra $\widetilde{\mathcal{C}}(A_S)$ of cyclic words in $A_{\mathcal{X}}$ by the subspace generated by the elements $(\sum_{s \in S} X_s)^n$, $n \geq 1$.

Theorem 5.4 a) The space $\tilde{\mathcal{C}}(I_{\mathcal{X}})$ is a Lie algebra ideal in $\tilde{\mathcal{C}}(A_{\mathcal{X}})$.
b) The element $(\sum_{s \in S} X_s)^n$ is in the center of the Lie algebra $\tilde{\mathcal{C}}(A_S)$.
c) Let us choose an element $x_0 \in \mathcal{X}$. Then there is a canonical isomorphism of Lie algebras

$$\frac{\tilde{\mathcal{C}}(A_{\mathcal{X}})}{\tilde{\mathcal{C}}(I_{\mathcal{X}})} \xrightarrow{=} \bar{\mathcal{C}}(A_S) \xrightarrow{\bar{\kappa}} \text{ODer}^{SS}(\bar{A}_{\mathcal{X}}) \quad (125)$$

Proof. Take cyclic words $A = \mathcal{C}\left((\sum_{x \in \mathcal{X}} X_x) \cdot A_1\right) \in \tilde{\mathcal{C}}(I_{\mathcal{X}})$ and $B \in \tilde{\mathcal{C}}(A_{\mathcal{X}})$. Then $[A, B]$, by the very definition of the commutator in the Lie algebra of cyclic words, is a sum of the terms corresponding to pairs (a generator in A , a generator in B). The pairs (a generator in A_1 , a generator in B) clearly produce an element of the subspace $I_{\mathcal{X}}$. It is easy to see that the sum of the terms corresponding to the remaining pairs $(\sum_{x \in \mathcal{X}} X_x, \text{a generator in } B)$ is zero: this is very similar to the proof of the fact that a cyclic word provide a derivation killing $\sum_{s \in S} X_s$, and the same kind of argument proves b). So $\tilde{\mathcal{C}}(I_{\mathcal{X}})$ is a Lie ideal in $\tilde{\mathcal{C}}(A_{\mathcal{X}})$, and $\bar{\mathcal{C}}(A_S)$ is a Lie algebra.

The isomorphism (124) shows that one obviously has an isomorphism of vector spaces

$$\frac{\mathcal{C}(A_{\mathcal{X}})}{\mathcal{C}(I_{\mathcal{X}})} \xrightarrow{=} \mathcal{C}(A_S)$$

Passing to the $\tilde{\mathcal{C}}$ -quotients on the left we get an isomorphism of vector spaces

$$\frac{\tilde{\mathcal{C}}(A_{\mathcal{X}})}{\tilde{\mathcal{C}}(I_{\mathcal{X}})} \xrightarrow{=} \bar{\mathcal{C}}(A_S)$$

Notice that the element $X_{x_0}^n$ is zero in $\tilde{\mathcal{C}}(A_{\mathcal{X}})$ but it is not zero in $\tilde{\mathcal{C}}(A_S)$. So we had to define the quotient $\bar{\mathcal{C}}(A_S)$. Moreover, since any element of $\bar{A}_{\mathcal{X}}$ can be written modulo $I(\mathcal{X})$ as element of A_S (i.e. we use again (124)) this isomorphism commutes with the Lie brackets.

The second equality in (125) follows from the description of $\text{Ker} p_{x_0}$ and proposition 5.1. The theorem is proved.

We define the group $\text{Out}^{SSE} \pi_1^{(l)}(X_N)$ of outer semi-special *equivariant* automorphisms of $\pi_1^{(l)}(X_N)$ as the invariants of the action of μ_N on $\text{Out}^{SS} \pi_1^{(l)}(X_N)$, and proceed similarly in the other cases.

If $\mathcal{X}_G := \{0\} \cup G \cup \{\infty\}$, $x_0 = \{\infty\}$, theorem 5.4 provides a description of the Lie algebra $\text{ODer}^{SSE} A(G)$ of outer semi-special *equivariant* derivations of $A(G)$:

$$\left(\frac{\tilde{\mathcal{C}}(A_{\mathcal{X}_G})}{\tilde{\mathcal{C}}(I_{\mathcal{X}_G})} \right)^G \xrightarrow{=} \text{ODer}^{SSE}(\bar{A}_{\mathcal{X}_G}) \quad (126)$$

Combining this with theorem 5.2b) we get a description of the Lie algebra $\text{ODer}^{SSE}(\bar{\mathcal{L}}_{\mathcal{X}_G}) \subset \text{ODer}^{SSE}(\bar{A}_{\mathcal{X}_G})$.

7. The distribution relations. Recall that $X_N = \mathbb{P}^1 \setminus \{0, \mu_N, \infty\}$ and there is canonical isomorphism $\mathrm{Gr}^W \mathbb{L}(X_M, v_\infty) = L(\mu_N)$. The maps

$$i_N : X_{NM} \hookrightarrow X_M, \quad z \mapsto z, \quad m_N : X_{NM} \longrightarrow X_M, \quad z \mapsto z^N$$

induce the surjective Lie algebra homomorphisms

$$i_{N*} : L(\mu_{MN}) \longrightarrow L(\mu_M); \quad m_{N*} : L(\mu_{MN}) \longrightarrow L(\mu_M)$$

given on the generators by

$$\begin{aligned} i_{N*} : Y &\longrightarrow Y, \quad X_\zeta \longrightarrow \begin{cases} 0 & \zeta \notin \mu_M \\ X_\zeta & \zeta \in \mu_M \end{cases} \\ m_{N*} : Y &\longrightarrow NY, \quad X_\zeta \longrightarrow X_{\zeta^N} \end{aligned}$$

Lemma 5.5 *Elements of $\mathrm{ODer}^{SSE} L(\mu_{MN})$ preserve $\mathrm{Ker}(i_{N*})$ and $\mathrm{Ker}(m_{N*})$.*

Proof. $\mathrm{Ker}(i_{N*})$ is generated by the elements X_ζ where $\zeta \notin \mu_N$. Since any semi-special derivation preserves conjugacy class of X_ζ , it in particular preserves the ideal generated by such X_ζ .

The statement about m_N follows from the fact that equivariant derivations, by their very definition, commute with the natural action of the group G on $L(\mu_G)$. Namely, let $\mathcal{D} \in \mathrm{Der}^E L(\mu_{MN})$. Then $\xi_*(\mathcal{D}(X_\zeta)) = \mathcal{D}(X_{\xi\zeta})$, and so m_{N*} sends $\mathcal{D}(X_\zeta)$ and $\mathcal{D}(X_{\xi\zeta})$ to the same element. The lemma is proved.

It follows that there are well defined homomorphisms

$$\tilde{i}_N, \tilde{m}_N : \mathrm{ODer}^{SSE} L(\mu_{MN}) \longrightarrow \mathrm{ODer}^{SSE} L(\mu_M) \quad (127)$$

uniquely characterized by the property that for any element $l \in L(\mu_{MN})$ and a derivation $D \in \mathrm{ODer}^{SSE} L(\mu_{MN})$ one has

$$f_* \circ D(l) = \tilde{f}(D) \circ f(l); \quad f = i_N, m_N \quad (128)$$

The same arguments provide us with homomorphisms

$$\bar{i}_N, \bar{m}_N : \mathrm{Out}^{SSE} L^{(l)}(\mu_{MN}) \longrightarrow \mathrm{Out}^{SSE} L^{(l)}(\mu_M) \quad (129)$$

Recall the maps

$$\psi_N^{(l)} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty N})) \longrightarrow \mathrm{Out}^{SSE} L^{(l)}(\mu_N)$$

They together with homomorphisms (129) provide the following two commutative diagrams (similar to (128)), where j_M is the natural inclusion:

$$\begin{array}{ccc} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty MN})) & \xrightarrow{\psi_{MN}^{(l)}} & \mathrm{Out}^{SSE} L^{(l)}(\mu_{MN}) \\ \downarrow j_M & & \downarrow \bar{i}_N \\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty M})) & \xrightarrow{\psi_M^{(l)}} & \mathrm{Out}^{SSE} L^{(l)}(\mu_M) \end{array}$$

and

$$\begin{array}{ccc} \text{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty MN})\right) & \xrightarrow{\psi_{MN}^{(l)}} & \text{Out}^{SSE} L^{(l)}(\mu_{MN}) \\ \downarrow j_M & & \downarrow \overline{m}_N \\ \text{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty M})\right) & \xrightarrow{\psi_M^{(l)}} & \text{Out}^{SSE} L^{(l)}(\mu_M) \end{array}$$

The composition $\psi_M^{(l)} \circ j_M$ does not depend on the choice of the maps $\overline{i}_N, \overline{m}_N$. Passing to the Lie algebras we get

$$\text{Gr}^W \mathcal{G}_{NM}^{(l)} \subset \text{Ker}(\tilde{i}_N - \tilde{m}_N) \otimes \mathbb{Q}_l \quad (130)$$

The maps i_{N*}, m_{N*} provide the linear maps

$$i'_{N*}, m'_{N*} : \mathcal{C}(A(\mu_{MN}))^{\mu_{MN}} \longrightarrow \mathcal{C}(A(\mu_M))^{\mu_N}$$

We set $i''_N := i'_{N*}$ and $m''_N := N^{-1}m_{N*}$.

Lemma 5.6 *The maps i''_N, m''_N preserve the Lie brackets.*

Proof. Direct check.

Warning. The map m''_N is a Lie algebra morphism only when restricted to the subspace of G -invariant cyclic words.

Since

$$i''_N - m''_N : Y^k \longmapsto (1 - N^{k-1})Y^k, \quad \sum_{g \in \mu_{MN}} X_g^k \longmapsto 0 \quad (131)$$

the identification $\text{Der}^{SE} A(\mu_G) = \tilde{\mathcal{C}}(A(\mu_G))^G$ leads to the Lie algebra maps

$$\text{Der}^{SE} A(\mu_{MN}) \longrightarrow \text{Der}^{SE} A(\mu_M)$$

Then they restrict to the maps

$$i'_N, m'_N : \text{Der}^{SE} L(\mu_{MN}) \longrightarrow \text{Der}^{SE} L(\mu_M) \quad (132)$$

Proposition 5.7 *The distribution relations are equivalent to the conditions*

$$\text{Gr}^W \mathcal{G}_N^{(l)} \subset \text{Ker}(i'_L - m'_L) \quad \text{for each integer } L|N \quad (133)$$

Proof. An element of $\text{Der}^{SE} L(\mu_N)$ is given by expression

$$\sum a(g_0 : g_1 : \dots : g_m)_{n_0, \dots, n_m} \cdot \kappa_{\mu_N} \mathcal{C}(X_{g_0} Y^{n_0-1} \dots X_{g_m} Y^{n_m-1}) \quad (134)$$

The sum is over cyclic words. The a 's are coefficients in \mathbb{Q} . They satisfy the relation $a(g_0 : \dots : g_m)_* = a(hg_0 : \dots : hg_m)_*$ expressing the fact that the cyclic word (134) is G -invariant. It is in $\text{Ker}(i'_L - m'_L)$ if and only if

$$a(g_0 : g_1 : \dots : g_m)_{n_0, \dots, n_m} = L^{-1} L^{w-m} \sum_{h_i^L = g_i} a(h_0 : h_1 : \dots : h_m)_{n_0, \dots, n_m} \quad (135)$$

except the relation

$$a(g : g) = L^{-1} \sum_{h_i^L = g} a(h_0 : h_1) \quad (136)$$

for $m = 1$ and $g_0 = g_1$. This is precisely the distribution relations. Here is the origin of the exception. An element

$$\sum a(g_0 : g_1 : \dots : g_m)_{n_0, \dots, n_m} \cdot \mathcal{C}(X_{g_0} Y^{n_0-1} \dots X_{g_m} Y^{n_m-1})$$

is in $\text{Ker}(i_L'' - m_L'')$ if and only if equations (135) are valid. Then to get the maps (151) we first restrict to the subspace of those cyclic words which provide derivations of the Lie algebra $L(\mu_N)$, and then kill the elements by Y^2 and $\sum_{g \in G} X_g^2$. Therefore, thanks to (131), the kernel is increased by a one dimensional subspace: we add the element projected by $i_L'' - m_L''$ onto $\sum_{g \in \mu_{N/L}} X_g^2$. The equation we thus removed is exactly (136). Indeed, this equation just means that the $\sum_{g \in \mu_{N/L}} X_g^2$ component of $(i_L'' - m_L'')(134)$ is zero, which we no longer require. The proposition is proved.

One has

$$i_L' - m_L' : (Y + \sum_{g \in \mu_N} X_g)^2 \mapsto (1 - L)(Y + \sum_{g \in \mu_{N/L}} X_g)^2$$

So killing the elements $(Y + \sum X_g)^2$ we get well defined maps

$$\tilde{i}_L', \tilde{m}_L' : \text{ODer}^{SSE} L(\mu_N) \longrightarrow \text{ODer}^{SSE} L(\mu_{N/L})$$

and moreover, since $1 - L \neq 0$, we have $\text{Ker}(i_L' - m_L') = \text{Ker}(\tilde{i}_L' - \tilde{m}_L')$.

Lemma 5.8 *One has $\tilde{i}_L = i_L'$ and $\tilde{m}_L = m_L'$*

Proof. One checks that each of the maps satisfy condition (128). Since this condition characterizes these maps the lemma follows.

Now the distribution relations follow from (130), proposition 5.7 and lemma 5.8.

Proof of theorem 2.2. It follows immediately from the inversion relation, which just has been proved as $N = -1$ case of the distribution relations. The argument is identical with the one given in the proof of corollary 4.2.

8. Conjecture 1.1. To complete the proof of conjecture 1.1 it remains to prove that $\text{Gr}\mathcal{G}_{\bullet\bullet}^{(l)}(\mu_N)$ lies in the subspace of $\text{GrDer}_{\bullet\bullet}^{SE} L(\mu_N)$ defined by the power shuffle relations (62). So these relations provide the most nontrivial constraints on the image of the Galois group. This is rather amazing since they are the simplest relations on the level of functions. Unfortunately I do not know a good motivic proof of them. There exists, however, an indirect argument: relations (62) hold for functions \Rightarrow for the corresponding framed mixed Hodge

structures \Rightarrow for the corresponding mixed Tate motives over $\mathbb{Q}(\zeta_N) \Rightarrow$ valid for their l -adic realization. The detailed exposition will appear in [G4]. In s. 7.3-7.4 I prove certain special cases of conjecture 1.1 relevant to our situation which can be obtained by some ad hoc methods.

6 Cohomology of some discrete subgroups of $GL_2(\mathbb{Z})$ and $GL_3(\mathbb{Z})$

1. The general scheme. Let G be a reductive group over \mathbb{Q} . Denote by \mathbb{A}_f the ring of finite adels for \mathbb{Q} . Choose a finite index subgroup $K_f \subset G(\widehat{\mathbb{Z}})$. Let $Z \subset G$ be the maximal split torus in the center of G , and $Z^0(\mathbb{R})$ the connected component of identity of its \mathbb{R} -valued points. Denote by K'_∞ the connected component of the maximal compact subgroup of the derived group $G'(\mathbb{R})$. Set

$$K_\infty^{\min} := K'_\infty \times Z^0(\mathbb{R})$$

$$K_\infty^{\max} := \text{maximal compact subgroup of } G(\mathbb{R}) \times Z^0(\mathbb{R})$$

Choose a subgroup K_∞ sitting in between:

$$K_\infty^{\min} \subset K_\infty \subset K_\infty^{\max}$$

One defines the modular variety corresponding to a given choice of the subgroups K_f and K_∞ as follows:

$$S_{K_\infty \times K_f}^G = G(\mathbb{Q}) \backslash \left(G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f \right) \quad (137)$$

In general it is, of course, not a variety in the algebraic geometry sense, and not even a manifold, only an orbifold. A rational representation $\rho : G \otimes_{\mathbb{Q}} \overline{\mathbb{Q}} \rightarrow GL(V)$ provides a local system on the orbifold (137):

$$\mathcal{L}_V := \left(G(\mathbb{R})/K_\infty \times G(\mathbb{A}_f)/K_f \right) \times_{G(\mathbb{Q})} V$$

Example. $G = GL_m$, $K_\infty^{\max} = O_m \cdot \mathbb{R}_+^*$, $K_\infty^{\min} = SO_m \cdot \mathbb{R}_+^*$. The space $GL_m(\mathbb{R})/K_\infty^{\max}$ is identified with positive definite symmetric $m \times m$ matrices. The group $GL_m(\mathbb{R})$ acts on it by $X \mapsto AXA^t$. The symmetric space is

$$\mathbb{H}_m = GL_m(\mathbb{R})/K_\infty^{\max} \cdot \mathbb{R}_{>0}^* = \{> 0 \text{ definite symmetric } m \times m \text{ matrices}\} / \mathbb{R}_+^*$$

One defines

$$\Gamma := GL_m(\mathbb{Q}) \cap K_f \subset GL_m(\mathbb{Z})$$

If Γ is torsion free then

$$H^*(\Gamma, V) = H^*(S_{K_\infty \times K_f}^G, \mathcal{L}_V)$$

If Γ contains a finite index torsion free subgroup $\tilde{\Gamma}$ then $H^*(\Gamma, V) = H^*(\tilde{\Gamma}, V)^{\Gamma/\tilde{\Gamma}}$.

We will use the shorthand S_Γ for the modular variety corresponding to the subgroup Γ . The Borel-Serre compactification \overline{S}_Γ is a compact manifold with corners. The boundary $\partial\overline{S}_\Gamma$ is a topological manifold. It is a disjoint union of faces which correspond bijectively to the Γ -conjugacy classes of proper parabolic subgroups of G defined over \mathbb{Q} . The closure of the face corresponding to a parabolic subgroup P is called a strata and denoted $\partial_P S$. There is a natural restriction map

$$H^*(S_\Gamma, \mathcal{L}_V) \xrightarrow{\text{Res}} H^*(\partial\overline{S}_\Gamma, \mathcal{L}_V)$$

The cohomology at infinity $H_{\text{inf}}^*(\partial\overline{S}_\Gamma, \mathcal{L}_V)$ are, by definition, the image of this map. The computation of these groups is an important step in understanding of the cohomology of \mathcal{L}_V . To carry it out one should determine first the cohomology of the restriction of \mathcal{L}_V to the Borel-Serre boundary $\partial\overline{S}_\Gamma$.

Pick a rational parabolic subgroup P . Let U_P be its unipotent radical and $M_P := P/U_P$ the Levi quotient. Denote by \mathcal{N}_P the Lie algebra of U_P . Take the subgroup $P(\mathbb{R}) \cap K_\infty$ and project it to $K_\infty^M \subset M(\mathbb{R})$. In our case $P(\mathbb{A}_f) \cdot K_f = G(\mathbb{A}_f)$. Then the cohomology $H^*(\partial_P S, \mathcal{L}_V)$ of the restriction of the local system \mathcal{L}_V to the boundary strata $\partial_P S$ are calculated using the Leray spectral sequence whose E_2 -term looks as follows:

$$H^p(S_{K_\infty^M}^M, H^q(N_P(\mathbb{Z}), V)) \Rightarrow H^{p+q}(\partial_P S, \mathcal{L}_V) \quad (138)$$

One has

$$H^q(N_P(\mathbb{Z}), V) = H^q(\mathcal{N}_P, V)$$

The first step of the calculation of the cohomology at infinity of a subgroup Γ is the calculation of these groups for all of the strata.

The Kostant theorem. Let P be a rational parabolic subgroup of a reductive group G . Fixing Cartan and Borel subgroups $H(\mathbb{C}) \subset B(\mathbb{C})$ with $H(\mathbb{C}) \subset M_P(\mathbb{C})$ we get a system of positive roots and the Weyl group W . Denote by W_P the Weyl group for M_P with $W_P \subset W$. It is known that in each coset class $W_P \backslash W$ there is a unique element of minimal possible length. Denote by W_P^1 the set of the minimal length representatives for $W_P \backslash W$. For any dominant weight μ denote by L_μ the irreducible representation of the group $M_P(\mathbb{C})$ with the highest weight μ . Let ρ be the half of the sum of the positive roots for G .

Theorem 6.1 ([K], Theorem 5.14). *The $M_P(\mathbb{C})$ -representation $H^*(\mathcal{N}_P, V)$ is algebraic and is given by*

$$H^*(\mathcal{N}_P, V) = \bigoplus_{\omega \in W_P^1} L_{\omega(\lambda+\rho)-\rho}[-l(\omega)]$$

where $l(\omega)$ is the length of ω , and $[-l(\omega)]$ indicates that the module appears in degree $l(\omega)$. If V is defined over \mathbb{Q} , then so is $H^*(\mathcal{N}_P, V)$.

Let H be the maximal Cartan subgroup of GL_m given by the diagonal matrices. Denote by $[n_1, \dots, n_m]$ the character of H given by $(t_1, \dots, t_m) \longrightarrow t_1^{n_1} \cdot \dots \cdot t_m^{n_m}$. Let P be a parabolic subgroup containing H and M_P the Levi quotient of P . Then H is a maximal torus for M_P . Denote by $V_{[n_1, \dots, n_m]}^P$ or simply $V_{[n_1, \dots, n_m]}$ the representation with the highest weight $[n_1, \dots, n_m]$ of M_P .

Example. The weights ρ for GL_2 and GL_3 are given by

$$\rho = [1/2, -1/2]; \quad \rho = [1, 0, -1];$$

2. Cohomology of $GL_2(\mathbb{Z})$. The cohomology $H^*(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2)$ vanish for odd w . Indeed, if w is odd the central element $\text{diag}(-1, -1)$ acts by -1 in $S^{w-2}V_2$. So we will assume that $w - 2$ is even. Clearly

$$H^0(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) = 0$$

For any GL_2 -module V over \mathbb{Q} the Hochschild-Serre spectral sequence corresponding to the exact sequence

$$0 \longrightarrow SL_2(\mathbb{Z}) \longrightarrow GL_2(\mathbb{Z}) \xrightarrow{\det} \{\pm 1\} \longrightarrow 0$$

gives

$$H^*(GL_2(\mathbb{Z}), V) = H^*(SL_2(\mathbb{Z}), V)^+$$

Here $+$ is the invariants under the involution provided by conjugation by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. There is an exact sequence ([Sh])

$$0 \longrightarrow H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-2}V_2) \longrightarrow H^1(SL_2(\mathbb{Z}), S^{w-2}V_2) \xrightarrow{\text{Res}} H_{\text{inf}}^1(SL_2(\mathbb{Z}), S^{w-2}V_2) \longrightarrow 0$$

It admits a natural splitting as a module over the Hecke algebra.

Let us compute the cohomology of the restriction of the local system $\mathcal{L}_{S^{w-2}V_2}$ to the boundary strata $\partial_B S$ corresponding to the upper triangular Borel subgroup $B \subset GL_2$. In this case $M_B = T$ is the diagonal torus and $K_\infty^M = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$, $\xi_i = \pm 1$. We need to compute

$$H^p(K_\infty^M, H^q(\mathcal{N}_B, S^{w-2}V_2 \otimes \varepsilon_2))$$

We have $S^{w-2}V_2 \otimes \varepsilon_2 = L_{[w-1, 1]}$ and

$$H^q(\mathcal{N}_B, S^{w-2}V_2 \otimes \varepsilon_2) = \begin{cases} L_{[w-1, 1]} & q = 0 \\ L_{[0, w]} & q = 1 \end{cases}$$

Therefore $H^0 = 0$ (since $w - 1$ is odd), and

$$H^1(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) = H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-2}V_2)^- \oplus H_{\text{inf}}^1(SL_2(\mathbb{Z}), S^{w-2}V_2)$$

It is known that the $+$ and $-$ parts of $H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-2}V_2)$ are of the same dimension which, thanks to the Eichler-Shimura isomorphism, coincides with the dimension of the space of the holomorphic weight w cusp forms on $SL_2(\mathbb{Z})$.

2. Cohomology of $GL_3(\mathbb{Z})$ with coefficients in $S^{w-3}V_3$. The main result is

Theorem 6.2

$$H^i(GL_3(\mathbb{Z}), S^{w-3}V_3) = \begin{cases} \mathbb{Q} & i = 0, w = 3 \\ H_{\text{cusp}}^1(GL_2(\mathbb{Z}), S^{w-2}V_3 \otimes \varepsilon_2) & i = 3 \\ 0 & \text{otherwise} \end{cases} \quad (139)$$

Proof. Since the central element $-\text{Id} \in GL_3$ acts by -1 on V_3 , and hence on $S^{w-3}V_3$ if $w-3$ is odd, we get $H^i(GL_3(\mathbb{Z}), S^{w-3}V_3) = 0$ if $w-3$ is odd. We will assume from now on that $w-3$ is even.

For the definition of cuspidal cohomology $H_{\text{cusp}}^*(\Gamma, V)$ which we use below see s. 1.3 of [LS].

Proposition 6.3 a) *The following sequence is exact*

$$0 \longrightarrow H_{\text{cusp}}^i(GL_3(\mathbb{Z}), S^{w-3}V_3) \longrightarrow H^i(GL_3(\mathbb{Z}), S^{w-3}V_3) \longrightarrow H_{\text{inf}}^i(GL_3(\mathbb{Z}), S^{w-3}V_3) \longrightarrow 0$$

$$b) H_{\text{cusp}}^*(GL_3(\mathbb{Z}), S^{w-3}V_3) = 0.$$

Proof. Let sl_3 be the Lie algebra of $SL_3(\mathbb{R})$. Let π be a unitary irreducible representation of $SL_3(\mathbb{R})$. Denote by H_π^∞ the space of C^∞ -vectors in π . Since for $w > 3$ $S^{w-3}V_3$ is not self dual (i.e. the Cartan involution $\theta : g \mapsto (g^{-1})^t$ transforms it to a non isomorphic representation) one has according to proposition 6.12 in chapter II of [BW]

$$H^*(sl_3, SO_3; S^{w-3}V_3 \otimes H_\pi^\infty) = 0$$

The proposition follows from the argumentation given in chapter 1 of [LS].

The boundary of the Borel-Serre compactification of our modular variety is a disjoint union of three faces, $e(P_1)$, $e(P_2)$ and $e(B)$ which correspond to the $GL_3(\mathbb{Q})$ -equivalence classes of the following proper parabolic \mathbb{Q} -subgroups:

$$P_1 := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}; \quad P_2 := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \quad B := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

Their nilradicals are denoted by N_1 , N_2 and N ; the corresponding Lie algebras are \mathcal{N}_1 , \mathcal{N}_2 and \mathcal{N} . We are going to calculate the groups (138).

Applying Kostant's theorem we get the following:

$$H^q(\mathcal{N}_1, S^{w-3}V_3) = \begin{cases} L_{[w-3,0,0]} & q = 0 \\ L_{[-1,w-2,0]} & q = 1 \\ L_{[-2,w-2,1]} & q = 2 \end{cases}$$

$$H^q(\mathcal{N}_2, S^{w-3}V_3) = \begin{cases} L_{[w-3,0,0]} & q = 0 \\ L_{[w-3,-1,1]} & q = 1 \\ L_{[-1,-1,w-1]} & q = 2 \end{cases}$$

$$H^q(\mathcal{N}, S^{w-3}V_3) = \begin{cases} L_{[w-3,0,0]} & q = 0 \\ L_{[-1,w-2,0]} \oplus L_{[w-3,-1,1]} & q = 1 \\ L_{[-1,-1,w-1]} \oplus L_{[-2,w-2,1]} & q = 2 \\ L_{[-2,0,w-1]} & q = 3 \end{cases}$$

Let

$$S_1 := \begin{pmatrix} \pm 1 & 0 \\ 0 & GL_2(\mathbb{Z}) \end{pmatrix}; \quad S_2 := \begin{pmatrix} GL_2(\mathbb{Z}) & 0 \\ 0 & \pm 1 \end{pmatrix}; \quad S := \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

If there is a ± 1 subgroup in one of these groups, S_i , which acts by -1 on a module V , then $H^i(S_i, V) = 0$ for all i . This simple remark shows that we get a nonzero contribution only the following cases:

- i) For the group S_1 : $L_{[w-3,0,0]}$ and $L_{[-2,w-2,1]}[-2]$.
- ii) For the group S_2 : $L_{[w-3,0,0]}$ and $L_{[-1,-1,w-1]}[-2]$.
- iii) For the group S : $L_{[w-3,0,0]}$ and $L_{[-2,0,w-1]}[-3]$.

We consider them now case by case, indicating the value (p, q) for the $E_2^{p,q}$ -term of the spectral sequence where they appear. Recall that $w - 3$ is even.

- i) For the strata $\partial_{P_1}S$ we have the following:

$$H^0(S_1, L_{[w-3,0,0]}) = L_{[w-3,0,0]}; \quad (p, q) = (0, 0) \quad (140)$$

and

$$\begin{aligned} H^1(S_1, L_{[-2,w-2,1]}) &= H^1(GL_2(\mathbb{Z}), S^{w-3}V_3 \otimes \varepsilon_2) = \\ H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^- &\oplus H_{\text{inf}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2); \quad (p, q) = (1, 2) \end{aligned} \quad (141)$$

The other cohomology groups are zero.

- ii) Further, for the strata $\partial_{P_2}S$

$$H^0(S_2, L_{[w-3,0,0]}) = \begin{cases} \mathbb{Q} & w = 3 \\ 0 & w > 3 \end{cases}; \quad (p, q) = (0, 0)$$

$$\begin{aligned} H^1(S_2, L_{[w-3,0,0]}) &= H^1(GL_2(\mathbb{Z}), S^{w-3}V_2) = \\ H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^+; &\quad (p, q) = (1, 0) \end{aligned}$$

The other cohomology groups vanish.

iii) Finally, for the strata $\partial_{P_2} S$ the only nonzero cohomology groups are

$$H^0(S, H^q(\mathcal{N}, S^{w-3}V_3)) = \begin{cases} \mathbb{Q} & q = 0 \\ \mathbb{Q} & q = 3 \\ 0 & \text{otherwise} \end{cases}$$

Since

$$\partial \overline{S} = \partial_{P_1} S \cup \partial_{P_2} S; \quad \partial_B S = \partial_{P_1} S \cap \partial_{P_2} S$$

we have the Mayer-Vietoris long exact sequence

$$\dots \longrightarrow H^*(\partial \overline{S}, \mathcal{L}_V) \longrightarrow H^*(\partial_{P_1} S, \mathcal{L}_V) \oplus H^*(\partial_{P_2} S, \mathcal{L}_V) \longrightarrow H^*(\partial_B S, \mathcal{L}_V) \longrightarrow \dots$$

It is easy to see that the differential in this complex maps isomorphically the group (140) to the group from the step iii) with $q = 0$.

Further it follows from the definitions that the differential maps isomorphically the group H_{inf}^1 from (141) onto the group from the step iii) with $q = 3$.

Therefore the boundary cohomology are

$$H^i(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3}) = \begin{cases} H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^+ & i = 1 \\ H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^- & i = 3 \end{cases}$$

To determine the subgroup $H_{\text{inf}}^i(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3})$ one usually uses the theory of Eisenstein cohomology classes originated by G. Harder. See [H6] for details in our case. However just in our case there is another approach. Namely, we will show in the next chapter that $H^3(GL_3(\mathbb{Z}), \mathcal{L}_{S^{w-3}V_3})$ contains the subgroup $H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^-$, (see (163)). The proof uses theorem 1.3 in [G3] and some simple arguments presented in the section 7.1 which are independent of the rest of the paper. Therefore

$$H_{\text{inf}}^3(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3}) = H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^- \quad (142)$$

It remains to show that H_{inf}^1 is zero. The Poincaré duality provides a nondegenerate pairing

$$H^1(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3}) \otimes H^3(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3^\vee}) \longrightarrow \mathbb{Q} \quad (143)$$

Employing the Cartan involution θ of the group GL_3 which transforms V_3 into V_3^\vee we get

$$H_{\text{inf}}^3(\partial \overline{S}, \mathcal{L}_{S^{w-3}V_3^\vee}) = H_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{w-3}V_2)^- \quad (144)$$

If $x \in H^1(S, \mathcal{L}_{S^{w-3}V_3})$ and $y \in H^3(S, \mathcal{L}_{S^{w-3}V_3^\vee})$ then $\text{Res}(x) \cup \text{Res}(y) = 0$. Therefore (142) together with the Poincaré pairing (143) imply that $\text{Res}(x) = 0$. The theorem is proved.

3. Cohomology of $\Gamma_1(3; p)$. Recall that if Γ is a torsion free subgroup of $SL_3(\mathbb{Z})$ then

$$H^*(\Gamma, \mathbb{Q}) = H^*(S_\Gamma, \mathbb{Q}) = H^*(\overline{S}_\Gamma, \mathbb{Q})$$

Lee and Schwermer derived in the section 1.8 of [LS] an exact sequence

$$0 \longrightarrow H_{\text{cusp}}^*(\Gamma, \mathbb{C}) \longrightarrow H^*(\overline{S}_\Gamma, \mathbb{C}) \longrightarrow H_{\text{inf}}^*(\overline{S}_\Gamma, \mathbb{C}) \longrightarrow 0$$

One has $H^q(S_\Gamma, \mathbb{Q}) = 0$ for $q > 3$. Further, $H^1(S_\Gamma, \mathbb{Q}) = 0$ by ch. 16 in [BMS]. The cuspidal cohomology satisfy the Poincaré duality. Therefore

$$\dim H_{\text{cusp}}^2(\Gamma_1(3; p), \mathbb{Q}) = \dim H_{\text{cusp}}^3(\Gamma_1(3; p), \mathbb{Q}) \quad (145)$$

Theorem 6.4 *Let p be a prime number. Then*

$$H_{\text{inf}}^i(\Gamma_1(3; p), \mathbb{Q}) = \begin{cases} \mathbb{Q} & q = 0 \\ 0 & q = 1, q \geq 4 \\ H_{\text{cusp}}^1(\Gamma(2; p), V_2) & q = 2 \\ H_{\text{cusp}}^1(\Gamma(2; p), \varepsilon_2) \otimes \mathbb{Q}^2 \oplus \mathbb{Q}^{\frac{p-3}{2}} & q = 3 \end{cases} \quad (146)$$

Proof. Let $\Gamma(m; p)$ be the full congruence subgroup of level p of $GL_m(\mathbb{Z})$. The cohomology of the subgroup $\Gamma(3; p)$ has been computed by Lee and Schwermer [LS]. Let me recall (and slightly correct) their result.

Set $SL_m^\pm(F_p) := GL_m(\mathbb{Z})/\Gamma(m; p)$. Here F_p is the finite field of order p . Then $H_{\text{inf}}^*(\overline{S}_{\Gamma(3m; p)}, \mathbb{Q})$ is a $SL_m^\pm(F_p)$ -module,

Consider the following subgroups in $SL_3^\pm(F_p)$:

$$\begin{aligned} \tilde{P}_1 &:= \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & \pm 1 \end{pmatrix} & \tilde{M}_1 &:= \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \\ \tilde{P}_2 &:= \begin{pmatrix} \pm 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} & \tilde{B} &:= \begin{pmatrix} \pm 1 & * & * \\ 0 & \pm 1 & * \\ 0 & 0 & \pm 1 \end{pmatrix} \end{aligned}$$

Denote by P'_1 , M'_1 and P'_2 the similar subgroups of $SL_3(F_p)$. Let $\tilde{\Gamma}_1(m; p)$ be the projection of $\Gamma_1(m; p)$ onto $SL_m^\pm(F_p)$. Denote by ξ'_i (resp. $\tilde{\xi}_i$) the nontrivial one dimensional representation of P'_i (resp. \tilde{P}_i) pulled from the one of the GL_2 -part of the Levi quotient given by the determinant. Denote by $[V]$ the isomorphism class of a representation V . Recall that V_2 is the standard two dimensional GL_2 -module.

Let $\widetilde{\text{St}}_3$ be the generalized Steinberg representation of $SL_3^\pm(F_p)$; it sits in the exact sequence

$$0 \longrightarrow \mathbb{Q} \longrightarrow \bigoplus_{i=1}^2 \text{Ind}_{\tilde{P}_i}^{SL_3^\pm(F_p)} \mathbb{Q} \longrightarrow \text{Ind}_{\tilde{B}}^{SL_3^\pm(F_p)} \mathbb{Q} \longrightarrow \widetilde{\text{St}}_3 \longrightarrow 0$$

It was proved in [LS], see s.2.5 there, that, considered as a $SL_3(F_p)$ -module,

$$[H_{\text{inf}}^q(\overline{S}_{\Gamma(3; p)}, \mathbb{Q})] =$$

$$= \begin{cases} \mathbb{Q} & q = 0 \\ 0 & q = 1, q \geq 4 \\ 2[\text{Ind}_{P'_1}^{SL_3(F_p)} \xi'_1] \oplus [\text{Ind}_{P'_1}^{SL_3(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), V_2)] & q = 2 \\ \oplus_{i=1}^2 [\text{Ind}_{P'_i}^{SL_3(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), \varepsilon_2)] \oplus [\tilde{\text{St}}_3] & q = 3 \end{cases} \quad (147)$$

In fact for $q = 3$ in [LS] appears the module $\oplus_{i=1}^2 [\text{Ind}_{P'_i}^{SL_3(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), \mathbb{Q})]$, but the P'_i -module $[H_{\text{cusp}}^1(\Gamma(2; p), \mathbb{Q})]$ should be changed to $[H_{\text{cusp}}^1(\Gamma(2; p), \varepsilon_2)]$. As abelian groups they are isomorphic. See also an argument in s. 7.7 below which shows that \mathbb{Q} has to be changed to ε_2 .

One can check, following the arguments in [LS], that, as a $SL_3^\pm(F_p)$ -module, $H_{\text{inf}}^q(\tilde{S}_{\Gamma(3;p)}, \mathbb{Q})$ in the nontrivial cases $q = 2, 3$ looks as follows:

$$[H_{\text{inf}}^q(\tilde{S}_{\Gamma(3;p)}, \mathbb{Q})] = \begin{cases} 2[\text{Ind}_{\tilde{P}_1}^{SL_3^\pm(F_p)} \xi_1] \oplus [\text{Ind}_{\tilde{P}_1}^{SL_3^\pm(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), V_2)]; & q = 2 \\ \oplus_{i=1}^2 [\text{Ind}_{\tilde{P}_i}^{SL_3^\pm(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), \varepsilon_2)] \oplus [\tilde{\text{St}}_3] & q = 3 \end{cases} \quad (148)$$

(We replaced SL_3 by SL_3^\pm and put ' by instead of tilde over P_i and ξ_i).

To compute $H_{\text{inf}}^*(\Gamma_1(3; p), \mathbb{Q})$ we need to find the invariants of the action of $\tilde{\Gamma}_1(3; p)$ on these $SL_3^\pm(F_p)$ -modules. We will use the following general statement.

Lemma 6.5 *Let $\rho : \tilde{P}_1 \longrightarrow \text{Aut}(W)$ be a representation trivial on the unipotent radical and ± 1 subgroup, and such that $H^0(SL_2^\pm(F_p), W) = 0$. Then*

$$\left(\text{Ind}_{\tilde{P}_1}^{SL_3^\pm(F_p)} W \right)^{\tilde{\Gamma}_1(3;p)} = W^{\tilde{\Gamma}_1(2;p)}$$

Proof. The left hand side is given by the space of W -valued functions $f(g)$ on $SL_3^\pm(F_p)$ satisfying the condition

$$f(\gamma g y) = \rho(y^{-1}) \cdot f(g), \quad \gamma \in \tilde{\Gamma}_1(3; p), \quad y \in \tilde{P}_1.$$

The coset $\tilde{\Gamma}_1(3; p) \backslash SL_3^\pm(F_p)$ is identified with the set of rows $X = (x_1, x_2, x_3) \in F_p^3 - 0$. The group \tilde{P}_1 acts on it from the right. So we need to determine the space of the W -valued functions $f(X)$ such that $f(Xy) = \rho(y^{-1})f(X)$. Any non zero vector (x_1, x_2, x_3) is $\tilde{\Gamma}_1(3; p)$ -equivalent to $(0, 1, 0)$ or $(0, 0, x)$. The stabilizer of $(0, 1, 0)$ is $\tilde{\Gamma}_1(2; p) \times \{\pm 1\}$. So $f(0, 1, 0) \in W^{\tilde{\Gamma}_1(2;p)}$. The stabilizer of $(0, 0, x)$ is $SL_2^\pm(F_p) \times \{\pm 1\}$. Since $H^0(SL_2^\pm(F_p), W) = 0$ we have $f(0, 0, x) = 0$. The lemma is proved.

Applying the lemma we get the following spaces of $\tilde{\Gamma}_1(3; p)$ -invariants:

- i) For $\text{Ind}_{\tilde{P}_1}^{SL_3^\pm(F_p)} \tilde{\xi}_1$ it is zero.
- ii) For $\text{Ind}_{\tilde{P}_1}^{SL_3^\pm(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), V_2)$ it is $H_{\text{cusp}}^1(\Gamma_1(2; p), V_2)$.

iii) For $\text{Ind}_{\tilde{P}_i}^{SL_3^\pm(F_p)} H_{\text{cusp}}^1(\Gamma(2; p), \varepsilon_2)$ it is $H_{\text{cusp}}^1(\Gamma_1(2; p), \varepsilon_2)$.

iv) The dimension of the space of $\tilde{\Gamma}_1(3; p)$ -invariants on $\widetilde{\text{St}}_3$ is $\frac{p-3}{2}$.

Argumentation. i) Indeed, $\xi(g) = -1$ for $g := \text{diag}(-1, 1, 1) \in \tilde{\Gamma}_1(3; p)$.

ii) and iii) are clear.

iv) It is a corollary of the following elementary statements, which are checked similarly to the proof of the lemma above:

$$\dim\left(\text{Ind}_{\tilde{B}}^{SL_3^\pm(F_p)} \mathbb{Q}\right)^{\tilde{\Gamma}_1(3; p)} = 3 \cdot \frac{p-1}{2}$$

$$\dim\left(\text{Ind}_{\tilde{P}_i}^{SL_3^\pm(F_p)} \mathbb{Q}\right)^{\tilde{\Gamma}_1(3; p)} = \frac{p-1}{2} + 1$$

and the trivial module \mathbb{Q} is, of course, $\tilde{\Gamma}_1(3; p)$ -invariant. So we get

$$3 \cdot \frac{p-1}{2} - 2\left(\frac{p-1}{2} + 1\right) - 1 = \frac{p-3}{2}$$

The theorem is proved.

7 Proofs of the theorems from section 2

1. The Soulé elements. Since $\text{Gr}\mathcal{G}_{\bullet, -1}^{(l)}$ is abelian we may identify it with the \mathbb{Q}_l -points of the corresponding algebraic group. So it makes sense to talk about projection of $\varphi^{(l)}(\sigma)$ on $\text{Gr}\mathcal{G}_{\bullet, -1}^{(l)}$. Let $\varphi_{-w, -1}^{(l)}(\sigma)$ be the component of this projection in $\text{Gr}\mathcal{G}_{-w, -1}^{(l)}$.

In [So] Soulé constructed for each integer $m > 1$ a $\text{Gal}(\mathbb{Q}(\zeta_{l^\infty})/\mathbb{Q})$ -homomorphism

$$\chi_m : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_{l^\infty}))^{ab} \longrightarrow \mathbb{Q}_l(m)$$

He proved that it is zero if and only if m is even. Let $I_{2m-1}(e : e)^\vee$ be the generator of $D_{-2m+1, -1}$ dual to $I_{2m-1}(e : e)$. It follows from the Key Lemma B in [Ih2] that

$$\varphi^{(l)}(\sigma) = \frac{(1 - l^{m-1})^{-1}}{(m-1)!} \chi_m(\sigma) \xi\left(I_{2m-1}(e : e)^\vee\right) \quad (m : \text{odd} \geq 3)$$

2. The depth 1 case. The distribution relations, proved in s. 4.7, in the depth one case just mean that

$$\text{Gr}\mathcal{G}_{-w, -1}^{(l)}(\mu_N) \subset \xi_{\mu_N}\left(D_{-w, -1}(\mu_N)\right) \otimes \mathbb{Q}_l$$

Easy classical arguments combined with Borel's theorem show that

$$\dim D_{-w, -1}(\mu_N) = \dim \text{Hom}(K_{2w-1}(S_N), \mathbb{Q})$$

For $w = 1$ it is a reformulation of the Bass theorem, and the general case is completely similar. From the motivic theory of classical polylogarithms at N -th roots of unity we get the following (see theorem 2.1):

$$\dim \mathrm{Gr} \mathcal{G}_{-w,-1}^{(l)}(\mu_N) = \dim \mathrm{Hom}(K_{2w-1}(S_N), \mathbb{Q})$$

Indeed, it has been proved in [BD], see also [HW], that motivic classical polylogarithms at N -th roots of unity provide classes generating $K_{2w-1}(S_N) \otimes \mathbb{Q}$; in particular the l -adic regulators of these classes has been computed. Denote by $H_1^{(-w)}$ the part of degree $-w$ in H_1 , and similarly $H_1^{(-w,-m)}$ the part of degree $-w$, depth $-m$ in H_1 . We conclude that

$$H_1^{(-w)}(\mathcal{G}_N^{(l)} / \mathcal{F}_{-2} \mathcal{G}_N^{(l)}) = \mathrm{Gr} \mathcal{G}_{-w,-1}^{(l)}(\mu_N) = \quad (149)$$

$$\mathrm{Hom}(K_{2w-1}(S_N), \mathbb{Q}_l) = \xi_{\mu_N}(D_{-w,-1}(\mu_N)) \otimes \mathbb{Q}_l \quad (150)$$

3. Description of the image of the Galois group: the depth two case.

Theorem 7.1 *a) Conjecture 1.1 is valid in the depth 2 case, i.e.*

$$\mathrm{Gr} \mathcal{G}_{\bullet, \geq -2}^{(l)}(\mu_N) \hookrightarrow \xi_{\mu_N}(D_{\bullet, \geq -2}(\mu_N)) \otimes \mathbb{Q}_l$$

b) Moreover

$$\mathrm{Gr} \mathcal{G}_{\bullet, -2}^{(l)}(\mu_N) = \xi_{\mu_N} \left(\frac{\mathcal{D}_{\bullet, -2}(\mu_N)}{\mathrm{Ker}(\delta : \mathcal{D}_{\bullet, -2}(\mu_N) \longrightarrow \Lambda^2 \mathcal{D}_{\bullet, -1}(\mu_N))} \right)^\vee$$

Proof. The part b) implies the part a). Since $\mathrm{Gr}^W \mathcal{G}_N^{(l)}$ is a quotient of the fundamental Lie algebra $L_{T\mathbb{Q}_l}(S_N)$ (see s. 2.7) one has

$$H_1^{(-w)}(\mathcal{G}_N^{(l)} / \mathcal{F}_{-3} \mathcal{G}_N^{(l)}) \subset \mathrm{Hom}(K_{2w-1}(S_N), \mathbb{Q}_l) \quad (151)$$

Therefore

$$H_1^{(-w,-2)}(\mathcal{G}_N^{(l)} / \mathcal{F}_{-3} \mathcal{G}_N^{(l)}) = 0 \quad (152)$$

Indeed, $[\mathcal{G}_N^{(l)}, \mathcal{G}_N^{(l)}] \subset \mathcal{F}_{-2} \mathcal{G}_N^{(l)}$, so a nontrivial depth -2 part of H_1 plus (149) will make the left hand side of (151) bigger then the right hand side.

If \mathcal{G} is a Lie algebra with a filtration \mathcal{F}_\bullet indexed by integers $n = -1, -2, \dots$, such that $\mathcal{F}_{-1} \mathcal{G} = \mathcal{G}$, then the Lie algebra

$$\mathcal{G} / \mathcal{F}_{-3} \mathcal{G} \quad \text{is isomorphic to the Lie algebra} \quad \mathrm{Gr}_{\geq -2}^{\mathcal{F}} \mathcal{G} := \mathrm{Gr}_{-1}^{\mathcal{F}} \mathcal{G} \oplus \mathrm{Gr}_{-2}^{\mathcal{F}} \mathcal{G}$$

In particular $\mathcal{G}_N^{(l)}/\mathcal{F}_{-3}\mathcal{G}_N^{(l)}$ is isomorphic to $\mathrm{Gr}\mathcal{G}_{\bullet,\geq-2}^{(l)}(\mu_N)$. Therefore thanks to (152) we have

$$H_1^{(-w,-2)}\left(\mathrm{Gr}\mathcal{G}_{\bullet,\geq-2}^{(l)}(\mu_N)\right) = 0 \quad (153)$$

This means that $\mathrm{Gr}\mathcal{G}_{\bullet,\leq-2}^{(l)}(\mu_N)$ is generated by $\mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)}(\mu_N)$.
Therefore

$$\begin{aligned} \mathrm{Gr}\mathcal{G}_{\bullet,-2}^{(l)}(\mu_N) &\stackrel{(153)}{=} [\mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)}(\mu_N), \mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)}(\mu_N)] \stackrel{(150)}{=} \\ \xi_{\mu_N}([D_{\bullet,-1}(\mu_N), D_{\bullet,-1}(\mu_N)]) &\subset \xi_{\mu_N}(D_{\bullet,-2}(\mu_N)) \end{aligned}$$

This is equivalent to the statement of the theorem. The theorem is proved.

Remark. If $N = 1$, or $N = p$ is a prime and $\bullet = -2$, we prove below that the space

$$\mathrm{Ker}\left(\delta : \mathcal{D}_{\bullet,-2}(\mu_N) \longrightarrow \Lambda^2 \mathcal{D}_{\bullet,-1}(\mu_N)\right)$$

is zero. It may not be zero otherwise. It would be interesting to construct explicitly the elements in the kernel.

Using theorem 7.1 one can show that if N is not prime then $\xi_{\mu_N}(D_{\bullet}^{\Delta}(\mu_N))$ could be bigger than $\mathrm{Gr}\mathcal{G}_{\bullet}^{(l)}(\mu_N)$.

4. The image of the Galois group for $m = 3, N = 1$ case

Theorem 7.2

$$\mathrm{Gr}\mathcal{G}_{-w,-3}^{(l)} \hookrightarrow \xi(D_{-w,-3}) \otimes \mathbb{Q}_l$$

Proof. When w is odd this follows from theorem 2.2. So we may assume that w is even. The same argumentation as in the proof of theorem 7.1 gives

$$H_1^{(-w,-3)}\left(\mathcal{G}_N^{(l)}/\mathcal{F}_{-4}\mathcal{G}_N^{(l)}\right) = 0 \quad (154)$$

However *a priori* the Lie algebra $\mathcal{G}_N^{(l)}/\mathcal{F}_{-4}\mathcal{G}_N^{(l)}$ may be different from $\mathrm{Gr}\mathcal{G}_{\bullet,\geq-3}^{(l)}(\mu_N)$.

Recall that $\mathrm{Gr}^W\mathcal{G}_N^{(l)}$ is non canonically isomorphic to $\mathcal{G}_N^{(l)}$. We are going to show that if $N = 1$ and w is odd the weight w , depth ≤ 3 parts of the standard cochain complexes of $\mathrm{Gr}^W\mathcal{G}^{(l)}$ and $\mathrm{Gr}\mathcal{G}_{\bullet,\bullet}^{(l)}$ are isomorphic. For the depth ≤ 2 parts this has been proved above. So the discrepancy between them can appear only if there are elements $x \in \mathcal{F}_3\mathrm{Gr}^W(\mathcal{G}^{(l)})^{\vee}$ such that

$$\delta(x) \in \mathcal{F}_1\mathrm{Gr}^W(\mathcal{G}^{(l)})^{\vee} \wedge \mathcal{F}_1\mathrm{Gr}^W(\mathcal{G}^{(l)})^{\vee}$$

Since $\mathcal{F}_1\mathrm{Gr}_p^W(\mathcal{G}^{(l)})^{\vee}$ can be non zero only if p is odd, the weight of $\delta(x)$ must be even. This contradicts to the assumption that w is odd. Therefore

$$H_1^{(-w,-3)}\left(\mathrm{Gr}\mathcal{G}_{\bullet,\geq-3}^{(l)}\right) = H_1^{(-w,-3)}\left(\mathcal{G}_N^{(l)}/\mathcal{F}_{-4}\mathcal{G}_N^{(l)}\right) = 0 \quad (155)$$

Therefore this together with the previous theorem implies that when w is odd $\mathrm{Gr}\mathcal{G}_{-w,\geq-3}^{(l)}$ is generated by iterated commutators of triples of elements of $\mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)}$. Since $\mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)} = \xi(D_{\bullet,-1}) \otimes \mathbb{Q}_l$ we conclude that $\mathrm{Gr}\mathcal{G}_{-w,\geq-3}^{(l)}$ coincides with the subspace generated by iterated commutators of triples of elements of $\xi(D_{\bullet,-1}) \otimes \mathbb{Q}_l$, and so belongs to $\xi(D_{\bullet,-3}) \otimes \mathbb{Q}_l$. The theorem is proved.

Let $\mathcal{L}_{\bullet,-k}^{(l)} \subset \mathrm{Gr}\mathcal{G}_{\bullet,-k}^{(l)}$ be the subspace generated by iterated commutators of k elements of $\mathrm{Gr}\mathcal{G}_{\bullet,-1}^{(l)}$. Then $\mathcal{L}_{\bullet,\geq-m}^{(l)} := \oplus_{k=1}^m \mathcal{L}_{\bullet,-k}^{(l)}$ is a bigraded Lie subalgebra of $\mathrm{Gr}\mathcal{G}_{\bullet,\geq-m}^{(l)}$.

Corollary 7.3 $\mathcal{L}_{\bullet,-m}^{(l)} = \mathrm{Gr}\mathcal{G}_{\bullet,-m}^{(l)}$ for $m = 2, 3$.

Remark. Denote by ε_2 the one dimensional GL_2 -module given by the determinant. We proved recently that, assuming conjecture 1.1 below, one has

$$\mathrm{Gr}\mathcal{G}_{-w,-4}^{(l)} / \mathcal{L}_{-w,-4}^{(l)} = H_{\mathrm{cusp}}^1(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) \otimes \mathbb{Q}_l$$

5. The cohomology of the Lie algebras $D_{\bullet\bullet}$ and $\widehat{D}_{\bullet\bullet}$ of depths 2, 3. Recall the isomorphism of bigraded Lie algebras

$$\widehat{D}_{\bullet\bullet}(G) = D_{\bullet\bullet}(G) \oplus \mathbb{Q}(-1, -1)$$

where $\mathbb{Q}(-1, -1)$ is a one dimensional Lie algebra of the bidegree $(-1, -1)$. So there is the decomposition of the standard cochain complex of $\widehat{D}_{\bullet\bullet}(G)$:

$$\Lambda^* \widehat{D}_{\bullet\bullet}(G) = \Lambda^* D_{\bullet\bullet}(G) \oplus \Lambda^* D_{\bullet\bullet}(G) \otimes \mathbb{Q}_{(1,1)} \quad (156)$$

It provides a canonical morphism of complexes

$$\partial_m : \Lambda_{(m)}^* \widehat{D}_{\bullet\bullet}(G) \longrightarrow \Lambda_{(m-1)}^* D_{\bullet\bullet}(G)[-1]$$

Surprisingly it is easier to describe the structure of the Lie algebra $\widehat{D}_{\bullet\bullet}(\mu_N)$, although the Galois-theoretic or motivic meaning of its $\mathbb{Q}_{(-1,-1)}$ -component is unclear: it should correspond to $\zeta(1)$, or, better, Euler's γ -constant, whatever it means.

Remark. i) Notice the classical formula

$$d \log \Gamma(1-z) = \sum_{m \geq 1} \zeta(m) z^{m-1} dz; \quad \text{where we set } \zeta(1) := \gamma$$

ii) One has

$$\gamma = - \int_0^\infty e^{-t} \log t dt = - \int_0^\infty \frac{dt}{t} \circ (e^{-t} dt)$$

(The iterated integral on the right has to be regularized). So the Euler γ -constant should be thought of as an “irregular” period of a motive.

It was proved in theorem 1.3 in [G3], see also lemma 2.3, that

$$H_{(w,2)}^i(\widehat{D}_{\bullet\bullet}) = H^{i-1}(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2), \quad i = 1, 2 \quad (157)$$

$$H_{(w,3)}^i(\widehat{D}_{\bullet,\bullet}) = H^i(GL_3(\mathbb{Z}), S^{w-3}V_3) = 0, \quad i = 1, 2, 3 \quad (158)$$

Theorem 7.4 *Denote by $H_{(w,m)}$ the weight w , depth m part of H . Then*

$$H_{(w,2)}^i(D_{\bullet\bullet}) = H_{\text{cusp}}^{i-1}(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2), \quad i = 1, 2 \quad (159)$$

$$H_{(w,3)}^i(D_{\bullet,\bullet}) = 0, \quad i = 1, 2, 3 \quad (160)$$

Proof. Consider the inclusion of complexes provided by the depth 2 part of (156):

$$\left(\mathcal{D}_{\bullet,2} \longrightarrow \Lambda^2 \mathcal{D}_{\bullet,1} \right) \hookrightarrow \mathcal{D}_{\bullet,2} \longrightarrow \Lambda^2 \widehat{\mathcal{D}}_{\bullet,1} \xrightarrow{\widehat{\partial}_2} \widehat{\mathcal{D}}_{\bullet,1} \quad (161)$$

Here $\widehat{\partial}_2$ is the composition of the map ∂_2 followed by the natural inclusion $\mathcal{D}_{\bullet,1} \hookrightarrow \widehat{\mathcal{D}}_{\bullet,1}$. The isomorphism μ from theorem 1.2 in [G3] is easily extended to an isomorphism between the complex $\mathbb{M}_{(2)}^* \otimes_{GL_2(\mathbb{Z})} S^{\bullet-2}V_2$ and the complex on the right of (161), which takes $\tau_{[1,2]} \mathbb{M}_{(2)}^* \otimes_{GL_2(\mathbb{Z})} S^{\bullet-2}V_2$ just to the subcomplex on the left. This together with lemma 2.3 prove (159).

Notice that the second summand $\mathcal{D}_{\bullet,1}[-1]$ in the decomposition (156) of the depth 2 part of $\Lambda^* \widehat{\mathcal{D}}_{\bullet\bullet}$ provides the Eisenstein part of the cohomology.

The depth 3 part of the decomposition (156) is

$$\left(\mathcal{D}_{\bullet,3} \xrightarrow{\delta} \mathcal{D}_{\bullet,2} \otimes \mathcal{D}_{\bullet,1} \xrightarrow{\delta} \Lambda^3 \mathcal{D}_{\bullet,1} \right) \oplus \left(\mathcal{D}_{\bullet,2} \xrightarrow{\delta} \Lambda^2 \mathcal{D}_{\bullet,1} \right) \quad (162)$$

Therefore we see that

$$H_{(w,3)}^3(\widehat{D}_{\bullet\bullet}) \text{ contains as a direct summand } H_{\text{cusp}}^1(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) \quad (163)$$

This was the last bit needed to prove theorem 6.2.

Now we can start using theorem 6.2. By theorem 1.3 in [G3] and theorem 6.2 we have

$$H_{(w,3)}^i(\widehat{D}_{\bullet\bullet}) \stackrel{1.3}{=} H^i(GL_3(\mathbb{Z}), S^{w-3}V_3) \stackrel{6.2}{=} \begin{cases} 0 & i = 1, 2 \\ H_{\text{cusp}}^1(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) & i = 3 \end{cases} \quad (164)$$

Therefore using (159) we see that the complex on the left of (162) is acyclic. This proves formula (160). Theorem 7.4 is proved.

6. Proofs of theorems 2.4, 2.6, and 2.10. Set $D_{\bullet\bullet}^{(l)} := D_{\bullet\bullet} \otimes \mathbb{Q}_l$. Applying to $\{e : e|t_0 : t_1\} = \sum_{n>0} I_n(e : e)(t_0 - t_1)^{n-1}$ the distribution relation (66) we get $\mathcal{D}_{-2w,-1} = 0$. Therefore

$$\mathcal{G}_{\bullet,-1}^{(l)} = D_{\bullet,-1}^{(l)}$$

Since $D_{\bullet\bullet}^{(l)}$ and $\mathcal{L}_{\bullet\bullet}^{(l)}$ are Lie subalgebras of $\text{GrDer}_{\bullet\bullet}^{SE} \mathbb{L}^{(l)}$ this implies that

$$\mathcal{L}_{\bullet\bullet}^{(l)} \hookrightarrow D_{\bullet\bullet}^{(l)} \quad (165)$$

Thus there are commutative diagrams where $[\cdot, \cdot]$ are the commutator maps:

$$\begin{array}{ccc} \Lambda^2 \mathcal{L}_{\bullet,-1}^{(l)} & \xrightarrow{=} & \Lambda^2 D_{\bullet,-1}^{(l)} \\ \downarrow [\cdot, \cdot] & & \downarrow [\cdot, \cdot] \\ \mathcal{L}_{\bullet,-2}^{(l)} & \hookrightarrow & D_{\bullet,-2}^{(l)} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{L}_{\bullet,-2}^{(l)} \otimes \mathcal{L}_{\bullet,-1}^{(l)} & \longrightarrow & D_{\bullet,-2}^{(l)} \otimes D_{\bullet,-1}^{(l)} \\ \downarrow [\cdot, \cdot] & & \downarrow [\cdot, \cdot] \\ \mathcal{L}_{\bullet,-3}^{(l)} & \hookrightarrow & D_{\bullet,-3}^{(l)} \end{array}$$

Lemma 7.5 $H_{(\bullet,m)}^1(D_{\bullet\bullet}) = 0$ for $m = 2, 3$.

Proof. Since $H^0(GL_2(\mathbb{Z}), S^{w-2}V_2 \otimes \varepsilon_2) = 0$ the lemma follows from formulas (159) and (160).

The lemma means that the right vertical arrows in the diagrams above are surjective. Therefore the left diagram shows that the map

$$\mathcal{L}_{\bullet,-2}^{(l)} \hookrightarrow D_{\bullet,-2}^{(l)}$$

is surjective, and hence is an isomorphism. Thus the top arrow in the right diagram is an isomorphism. Therefore the second diagram shows that the map

$$\mathcal{L}_{\bullet,-3}^{(l)} \hookrightarrow D_{\bullet,-3}^{(l)}$$

is also an isomorphism. Summarizing we have

$$\mathcal{L}_{\bullet,-2}^{(l)} = D_{\bullet,-2}^{(l)} \quad \text{and} \quad \mathcal{L}_{\bullet,-3}^{(l)} = D_{\bullet,-3}^{(l)} \quad (166)$$

Combining this with corollary 7.3 we get

$$\mathcal{G}_{\bullet,-2}^{(l)} = D_{\bullet,-2}^{(l)} \quad \text{and} \quad \mathcal{G}_{-w,-3}^{(l)} = D_{-w,-3}^{(l)}, \quad w \text{ is odd} \quad (167)$$

After this we reduced the study of the depth -2 and -3 quotients of the Lie algebras $\mathcal{G}_{\bullet\bullet}^{(l)}$ and $\widehat{\mathcal{G}}_{\bullet\bullet}^{(l)}$ to the study of the corresponding quotients of the Lie algebras $D_{\bullet\bullet}$ and $\widehat{D}_{\bullet\bullet}$, which were investigated in [G3]. Therefore theorems 2.4a), 2.6a), b) and 2.10 follow from theorems 1.2 in [G3], lemma 2.3, proposition 6.3b) and theorem 6.2 in [G3]. It remains to prove the formula from theorem 2.6c). Thanks to theorem 2.2 we may assume that w is odd.

Computation of the dimensions. i) Consider the generating series

$$a_m(x) := \sum_{w>0} \dim \Lambda_{(w)}^m(\mathcal{D}_{\bullet,1}) x^w$$

We know that $a_1(x) = \frac{x^3}{1-x^2}$. Thus one has

$$a_m(x) = \frac{x^{3+5+\dots+(2m+1)}}{\prod_{i=1}^m (1-x^{2i})}$$

In particular

$$a_2(x) = \frac{x^8}{(1-x^2)(1-x^4)}, \quad a_3(x) = \frac{x^{15}}{(1-x^2)(1-x^4)(1-x^6)}$$

ii) The ring of modular forms for $SL_2(\mathbb{Z})$ is generated by the Eisenstein series $E_4(z)$ and $E_6(z)$. By the Eichler-Shimura theorem the dimension of the space of modular forms for $SL_2(\mathbb{Z})$ coincides with $\dim H^1(GL_2(\mathbb{Z}), S^{w-2}V_2)$. Therefore

$$\sum_{w \geq 2} \dim H^1(GL_2(\mathbb{Z}), S^{w-2}V_2) x^w = \frac{1}{(1-x^4)(1-x^6)} - 1$$

iii) Let

$$d_m(x) := \sum_{w > 0} \dim \mathcal{D}_{w,m} x^w$$

Computing the Euler characteristic of the depth 2 part of the standard cochain complex of the Lie coalgebra $\mathcal{D}_{\bullet\bullet}$ and using theorem 7.4 we get

$$d_2(x) - a_2(x) = -\frac{1}{(1-x^4)(1-x^6)} + 1$$

Therefore using the formulas above we get

$$d_2(x) = \frac{x^8}{(1-x^2)(1-x^6)}$$

This formula is equivalent to theorems 2.4b).

iv) Computing the Euler characteristic of the depth 3 part of the standard cochain complex of the Lie coalgebra $\mathcal{D}_{\bullet\bullet}$ and using theorem 7.4 we get

$$d_3(x) - d_2(x) \cdot a_1(x) + a_3(x) = 0$$

Therefore using the formulas above we get

$$d_3(x) = \frac{x^{11}(1+x^2-x^4)}{(1-x^2)(1-x^4)(1-x^6)}$$

This formula is equivalent to theorems 2.6c).

These results also provide the computation of the right hand sides of the formulas appearing in theorems 1.4 and 1.5 in [G3].

These theorems, in particular, imply that the double shuffle relations provide a *complete* list of constraints on the Lie algebra of the image of the Galois group in $\text{Aut} \pi_1^{(l)}(\mathbb{P}^1 - \{0, 1, \infty\}, v_\infty)$ in the depths -2 and -3 .

7. Proofs of theorems 2.14 and 2.15. They are similar to the proofs in the subsection 3 above. Recall that p is a prime number. We will use shorthands like $D_{\bullet}^{(l)}(\mu_p)$ for the diagonal Lie algebra $D_{\bullet}^{\Delta}(\mu_p)^{(l)}$. By theorem 2.12

$$\text{Gr} \mathcal{L}_{-1}^{(l)}(\mu_p) = D_{-1}^{(l)}(\mu_p)$$

This implies that

$$\mathcal{L}_{\bullet}^{(l)}(\mu_p) \hookrightarrow D_{\bullet}^{(l)}(\mu_p) \quad (168)$$

Thus there are commutative diagrams

$$\begin{array}{ccc} \Lambda^2 \mathcal{L}_{-1}^{(l)}(\mu_p) & \xrightarrow{=} & \Lambda^2 D_{-1}^{(l)}(\mu_p) \\ \downarrow [\cdot, \cdot] & & \downarrow [\cdot, \cdot] \\ \mathcal{L}_{-2}^{(l)}(\mu_p) & \hookrightarrow & D_{-2}^{(l)}(\mu_p) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{L}_{-2}^{(l)}(\mu_p) \otimes \mathcal{L}_{-1}^{(l)}(\mu_p) & \longrightarrow & D_{-2}^{(l)}(\mu_p) \otimes D_{-1}^{(l)}(\mu_p) \\ \downarrow [\cdot, \cdot] & & \downarrow [\cdot, \cdot] \\ \mathcal{L}_{-3}^{(l)}(\mu_p) & \hookrightarrow & D_{-3}^{(l)}(\mu_p) \end{array}$$

Lemma 7.6 $H_{(m)}^1(D_{\bullet}(\mu_p)) = 0$ for $m = 2, 3$.

Proof. Thanks to decomposition (156) one has $H^1(\widehat{D}_{\bullet\bullet}(G)) = H^1(D_{\bullet\bullet}(G))$. This implies $H^1(\widehat{D}_{\bullet}(G)) = H^1(D_{\bullet}(G))$. Theorems 1.2, 6.1, 6.2 in [G3] provide the following crucial result:

$$H_{(w,2)}^i(\widehat{D}_{\bullet}(\mu_p)) = H^{i-1}(\Gamma_1(2; p), \varepsilon_2), = H^{i-1}(\Gamma_1(p), \mathbb{Q})^- \quad (169)$$

$$H_{(w,3)}^i(\widehat{D}_{\bullet}(\mu_p)) = H^i(\Gamma_1(3; p), \mathbb{Q}); \quad i = 1, 2, 3 \quad (170)$$

In particular

$$H_{(2)}^1(\widehat{D}_{\bullet}(\mu_p)) = H^0(\Gamma_1(2; p), \varepsilon_2) \quad H_{(3)}^1(\widehat{D}_{\bullet}(\mu_p)) = H^1(\Gamma_1(3; p), \mathbb{Q}) \quad (171)$$

Both groups are zero: for the first one it is clear, and for the second see ch. 16 in [BMS] or use Kazhdan's theorem. The lemma is proved.

So the right vertical arrows in the diagrams above are surjective. Using the same arguments as in s. 6.3 we get Lie algebra isomorphisms

$$\mathcal{L}_{\geq -m}^{(l)}(\mu_p) = D_{\geq -m}(\mu_p) \otimes \mathbb{Q}_l, \quad m = 2, 3 \quad (172)$$

After this theorems 2.14 follows from theorem 7.1 the results of [G3]. Similarly theorem 2.15 follows from (172) and the results of [G3].

These results imply that the double shuffle relations provide a *complete* list of constraints on the weight = depth part of the Lie algebra of the image of the Galois group in $\text{Aut}\pi_1^{(l)}(X_p, v_{\infty})$ in the depths -2 and -3 .

8. The Lie coalgebra $\mathcal{D}_\bullet^{\text{un}}(\mu_p)$. Recall that p is a prime number. Let us define a linear map $v_p : \mathcal{D}_\bullet(\mu_p) \longrightarrow \mathbb{Q}$ by

$$v_p : \mathcal{D}_m(\mu_p) \longmapsto 0 \quad \text{for } m > 1, \quad v_p : I_{1,1}(1 : \zeta_p^a) \longmapsto 1, \quad (\zeta_p^a \neq 1)$$

Since there is no distribution relations when p is prime the map v_p is well defined. Let $\mathcal{D}_\bullet^{\text{un}}(\mu_p) \hookrightarrow \mathcal{D}_\bullet(\mu_p)$ be the codimension 1 subspace $\text{Ker}(v_p)$

Proposition 7.7 $\mathcal{D}_\bullet^{\text{un}}(\mu_p)$ is a sub Lie coalgebra of $\mathcal{D}_\bullet(\mu_p)$.

Proof. Let V be a vector space and $f \in V^*$. There is a map

$$\partial_f : \Lambda^n V \longrightarrow \Lambda^{n-1} V; \quad \partial^2 = 0$$

$$v_1 \wedge \dots \wedge v_n \longmapsto \sum_{i=1}^n (-1)^{i-1} f(v_i) v_1 \wedge \dots \wedge \widehat{v_i} \wedge \dots \wedge v_n$$

In particular the map v_p provides a degree -1 maps

$$\partial_{v_p} : \Lambda^n \mathcal{D}_\bullet(\mu_p) \longrightarrow \Lambda^{n-1} \mathcal{D}_\bullet(\mu_p)$$

There is an exact sequence

$$0 \longrightarrow \Lambda_{(m)}^2 \mathcal{D}_\bullet^{\text{un}}(\mu_p) \hookrightarrow \Lambda_{(m)}^2 \mathcal{D}_\bullet(\mu_p) \xrightarrow{\partial_{v_p}} \mathcal{D}_{m-1}^{\text{un}}(\mu_p) \longrightarrow 0$$

So to prove the proposition we need to check that the composition

$$\mathcal{D}_m(\mu_p) \xrightarrow{\delta} \Lambda_{(m)}^2 \mathcal{D}_\bullet(\mu_p) \xrightarrow{\partial_{v_p}} \mathcal{D}_{m-1}(\mu_p)$$

is zero. To simplify notations we set $\{g_0 : \dots : g_m\} := I_{1,\dots,1}(g_0 : \dots : g_m)$. Then

$$\partial_{v_p} \circ \delta \{g_0 : \dots : g_m\} = \tag{173}$$

$$\partial_{v_p} \left(\sum_{i=0}^{m+1} \{g_{i+1} : g_{i+2} : \dots : g_{i-1}\} \wedge \{g_{i-1} : g_i\} + \{g_{i-1} : g_i\} \wedge \{g_i : g_{i+1} : \dots : g_{i-2}\} \right)$$

Since $\{g_{i-1} : g_i\} = 0$ if $g_{i-1} = g_i$ we calculate (173) as follows. We locate g_0, \dots, g_{m+1} cyclically on the circle. Say that a string in this cyclic word is a sequence of letters g_i, g_{i+1}, \dots, g_j such that $g_i = g_{i+1} = \dots = g_j$ and $g_{i-1} \neq g_i$, $g_j \neq g_{j+1}$. The cyclic word splits into a union of strings. For instance if all g_i 's are distinct, we get $m+1$ one element strings. The sum (173) equals to

$$\begin{aligned} & \sum_{\text{strings}} \partial_{v_p} \left(\{g_{i+1} : g_{i+2} : \dots : g_{i-1}\} \wedge \{g_{i-1} : g_i\} \right) + \\ & \sum_{\text{strings}} \partial_{v_p} \left(\{g_j : g_{j+1}\} \wedge \{g_{j+1} : \dots : g_{i-1} : g_i\} \right) \end{aligned}$$

The sum of the two terms corresponding to a given string is obviously zero, so (173) is zero. The proposition is proved.

It follows from proposition 7.7 that the map

$$\partial_{v_p} : \Lambda_{(m)}^* \mathcal{D}_\bullet(\mu_p) \longrightarrow \Lambda_{(m-1)}^* \mathcal{D}_\bullet(\mu_p)[-1]$$

is a homomorphism of complexes.

9. The depth 2 part of the cochain complex of $\mathcal{D}_\bullet^{\text{un}}(\mu_p)$ and the cuspidal cohomology of $\Gamma_1(2;p)$.

Theorem 7.8 *There is a canonical isomorphism of complexes*

$$\mathcal{D}_2(\mu_p) \longrightarrow \Lambda^2 \mathcal{D}_1^{\text{un}}(\mu_p) = \tau_{[1,2]} \left(\mathbb{M}_{(2)}^* \otimes_{\Gamma_1(2;p)} \mathbb{Q} \right)$$

Proof. This theorem is a version of the results proved in s. 3.5-3.6 in [G2]. Let us extend v_p to $\widehat{\mathcal{D}}_1(\mu_p)$ by putting it zero on $\mathbb{Q}_{(1,1)}$. We define a map

$$\Lambda^2 \widehat{\mathcal{D}}_1(\mu_p) \xrightarrow{\delta'} \mathcal{D}_1(\mu_p) \oplus \mathcal{D}_1(\mu_p)$$

by setting $\delta' := (-\partial, \partial + v_p)$, i.e.

$$\{1 : \zeta_p^\alpha\} \wedge \{1 : \zeta_p^\beta\} \longmapsto \begin{cases} 0 \oplus \{1 : \zeta_p^\beta\} - \{1 : \zeta_p^\alpha\} & \alpha, \beta \neq 0 \\ -\{1 : \zeta_p^\beta\} \oplus \{1 : \zeta_p^\alpha\} & \alpha = 0, \beta \neq 0 \end{cases}$$

Thus we get the following complex, placed in degrees $[1, 3]$:

$$\mathcal{D}_2(\mu_p) \xrightarrow{\delta} \Lambda^2 \widehat{\mathcal{D}}_1(\mu_p) \xrightarrow{\delta'} \mathcal{D}_1(\mu_p) \oplus \mathcal{D}_1(\mu_p) \quad (174)$$

Theorem 7.9 *The complex (174) is canonically identified with the complex $\mathbb{M}_{(2)}^* \otimes_{\Gamma_1(2;p)} \mathbb{Q}$.*

Proof. We start with a lemma describing more explicitly the rank 2 modular complex.

Lemma 7.10 *The double shuffle relations for $m = 2$ are equivalent to the dihedral relations:*

$$\langle v_0, v_1, v_2 \rangle = - \langle v_1, v_0, v_2 \rangle = - \langle v_0, v_2, v_1 \rangle = \langle -v_0, -v_1, -v_2 \rangle$$

Proof. The double shuffle relations in $\langle \cdot, \cdot, \cdot \rangle$ generators look as follows:

$$\langle v_0, v_1, v_2 \rangle + \langle v_1, v_0, v_2 \rangle = 0, \quad \langle v_1, v_2 - v_1, -v_2 \rangle + \langle v_2, v_1 - v_2, -v_1 \rangle = 0$$

Changing variables $u_0 = v_1, u_1 = v_2 - v_1, u_2 = -v_2$ we write the second of them as $\langle u_0, u_1, u_2 \rangle + \langle -u_2, -u_1, -u_0 \rangle = 0$. The lemma follows.

It follows from this lemma that the modular complex for $GL_2(\mathbb{Z})$ is canonically isomorphic to the following complex of left $GL_2(\mathbb{Z})$ -modules:

$$\mathbb{Z}[GL_2(\mathbb{Z})] \otimes_{D_2} \xi_2 \xrightarrow{\partial} \mathbb{Z}[GL_2(\mathbb{Z})] \otimes_{D_{1,1}} \xi_{1,1} \xrightarrow{\partial'} \mathbb{Z}[GL_2(\mathbb{Z})/\tilde{B}]$$

where $D_2 \subset GL_2(\mathbb{Z})$ is the order 12 dihedral subgroup,

$$D_{1,1} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix} \right\}, \quad \tilde{B} := \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}$$

ξ_2 is the character of D_2 given by the determinant, and $\xi_{1,1}$ is a nontrivial character of $D_{1,1}$ killing $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$. The differentials commute with the left action of the group $GL_2(\mathbb{Z})$ and so are determined by their action on the unit in GL_2 . They are given by

$$\partial : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\partial' : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The coset $\Gamma_1(2; p) \backslash GL_2(\mathbb{Z})$ is identified with the set of rows $\{(\alpha, \beta) \in F_p^2 - 0\}$. One can identify it with the set of non zero rows $\{(\alpha, \beta, \gamma)\}$ with $\alpha + \beta + \gamma = 0$. Then

$$M_{(2)}^1 = \mathbb{Z}[\Gamma_1(2; p) \backslash GL_2(\mathbb{Z})] \otimes_{D_2} \xi_2 = \frac{\mathbb{Z}[(\alpha, \beta, \gamma) \in F_p^3 - 0 \mid \alpha + \beta + \gamma = 0]}{\text{the dihedral relations}}$$

$$M_{(2)}^2 = \mathbb{Z}[\Gamma_1(2; p) \backslash GL_2(\mathbb{Z})] \otimes_{D_{1,1}} \xi_{1,1} = \frac{\mathbb{Z}[(\alpha, \beta) \in F_p^2 - 0]}{(\alpha, \beta) = -(\beta, \alpha) = (\pm\alpha, \pm\beta)}$$

$M_{(2)}^3 = \mathbb{Z}[\Gamma_1(2; p) \backslash GL_2(\mathbb{Z})/\tilde{B}] =$ the abelian group with the generators $[\beta, 0]$ and $[0, \beta]$, where $\beta \neq 0$, and the only relation is symmetry under $\beta \mapsto -\beta$.

The complement $X_1(p) - Y_1(p)$ consists of $p - 1$ cusps. The natural covering $X_1(p) \rightarrow X_0(p)$ is unramified of degree $(p - 1)/2$. There are two cusps on $X_0(p)$: the 0 and ∞ cusps. So there are $(p - 1)/2$ cusps over 0 and over ∞ . Under the identification with $\mathcal{D}_1(\mu_p) \oplus \mathcal{D}_1(\mu_p)$ they correspond to the summands $\mathcal{D}_1(\mu_p)$.

The desired isomorphism of complexes is defined by

$$(\alpha, \beta, \gamma) \mapsto \{\zeta_p^\alpha, \zeta_p^\beta, \zeta_p^\gamma\}$$

$$(\alpha, \beta) \mapsto \{\zeta_p^\alpha, \zeta_p^{-\alpha}\} \wedge \{\zeta_p^\beta, \zeta_p^{-\beta}\}$$

$$[\beta, 0] \oplus [0, \beta'] \longmapsto \{\zeta_p^\beta, \zeta_p^{-\beta}\} \oplus \{\zeta_p^{\beta'}, \zeta_p^{-\beta'}\}$$

To check that it is indeed an isomorphism of the vector spaces notice that, similarly to lemma 7.10, the only relations among the elements $\{\zeta_p^\alpha, \zeta_p^\beta, \zeta_p^\gamma\}$ are the dihedral symmetry. Notice also that $\{1, 1, 1\} = 0$ and the only distribution relation $0 = \{1, 1, 1\} = \sum_{\alpha, \beta} \{\zeta_p^\alpha, \zeta_p^\beta, \zeta_p^{-\alpha-\beta}\}$ follows from the skew symmetry.

Theorem 7.9 is proved. Theorem 7.8 follows immediately from theorem 7.9.

10. The depth 3 part of $\Lambda^* \mathcal{D}_\bullet^{\text{un}}(\mu_p)$.

Theorem 7.11

$$H_{(w,3)}^i(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) = \begin{cases} 0 & i = 1 \\ H_{\text{cusp}}^2(\Gamma_1(3; p), \mathbb{Q}) \oplus H_{\text{cusp}}^1(\Gamma_1(2; p), V_2) & i = 2 \\ H_{\text{cusp}}^3(\Gamma_1(3; p), \mathbb{Q}) & i = 3 \end{cases} \quad (175)$$

Proof. There is an exact sequence of complexes

$$\begin{array}{ccccccc} \mathcal{D}_3(\mu_p) & \xrightarrow{\delta} & \mathcal{D}_2(\mu_p) \otimes \mathcal{D}_1^{\text{un}}(\mu_p) & \xrightarrow{\delta} & \Lambda^3 \mathcal{D}_1^{\text{un}}(\mu_p) \\ \downarrow = & & \downarrow & & \downarrow \\ \mathcal{D}_3(\mu_p) & \xrightarrow{\delta} & \mathcal{D}_2(\mu_p) \otimes \mathcal{D}_1(\mu_p) & \xrightarrow{\delta} & \Lambda^3 \mathcal{D}_1(\mu_p) \\ & & \downarrow \partial_{v_p} & & \downarrow \partial_{v_p} \\ & & \mathcal{D}_2(\mu_p) & \xrightarrow{\delta} & \Lambda^2 \mathcal{D}_1^{\text{un}}(\mu_p) \end{array}$$

A choice of splitting $\mathbb{Q} \longrightarrow \mathcal{D}_1(\mu_p)$ of the map v_p provides a splitting of the middle complex into a direct sum of the top and bottom complexes. Taking into account canonical decomposition (156) we end up with

$$\Lambda_{(3)}^*(\widehat{\mathcal{D}}_\bullet(\mu_p)) = \Lambda_{(3)}^*(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) \oplus \Lambda_{(2)}^*(\mathcal{D}_\bullet^{\text{un}}(\mu_p))[-1] \oplus \Lambda_{(2)}^*(\mathcal{D}_\bullet(\mu_p))[-1]$$

Notice also a noncanonical splitting

$$\Lambda_{(2)}^*(\mathcal{D}_\bullet(\mu_p)) = \Lambda_{(2)}^*(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) \oplus \mathcal{D}_1^{\text{un}}(\mu_p)[-1]$$

Therefore we have

$$\Lambda_{(3)}^*(\widehat{\mathcal{D}}_\bullet(\mu_p)) = \Lambda_{(3)}^*(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) \oplus \Lambda_{(2)}^*(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) \otimes \mathbb{Q}^2[-1] \oplus \mathcal{D}_\bullet^{\text{un}}(\mu_p)[-2]$$

This implies that

$$H_{\text{cusp}}^1(\Gamma_1(2; p), \varepsilon_2) \otimes \mathbb{Q}^2 \oplus \mathbb{Q}^{\frac{p-3}{2}}$$

is a direct summand of $H_{\text{inf}}^3(\Gamma_1(3; p), \mathbb{Q})$. This has been known to us through theorem 6.4. On the other hand this gives an additional confirmation that the result of [LS] concerning $H_{\text{inf}}^3(\Gamma(3; p), \mathbb{Q})$ has to be corrected, as was explained in s. 6.3.

Using the relation (169)-(170) between the depth 2 and 3 pieces of the cohomology of the Lie coalgebra $\widehat{\mathcal{D}}_\bullet(\mu_p)$ and the cohomology of groups $\Gamma_1(2; p)$ and $\Gamma_1(3; p)$, and combining this with the description of the cohomology of the group $\Gamma_1(3; p)$ given in the previous chapter, in particular theorem 6.4, we arrive to the proof of theorem 7.11.

Remark. Here are some cycles in $H^2(\widehat{\mathcal{D}}_\bullet(\mu_p))$:

$$\{\zeta_p^\alpha, \zeta_p^{-\alpha}\} \otimes \{\zeta_p^\alpha, \zeta_p^\beta, \zeta_p^\gamma\} + \{\zeta_p^\beta, \zeta_p^{-\beta}\} \otimes \{\zeta_p^\beta, \zeta_p^\alpha, \zeta_p^\gamma\}$$

They probably generate H^2 , and one should be able to prove directly using these cycles that $H^2(\widehat{\mathcal{D}}_\bullet(\mu_p)) = H_{\text{cusp}}^1(\Gamma_1(2; p), V_2)$

11. Proof of corollary 2.16. Let us introduce the following notation. Let \mathcal{D}_\bullet be a \mathbb{Z}_+ -graded Lie algebra. Denote by $\chi_{(m)}(\mathcal{D}_\bullet)$ the Euler characteristic of the degree m part of the standard cochain complex of \mathcal{D}_\bullet .

Corollary 7.12 *For $m = 2, 3$ one has*

$$\chi_{(m)}(\mathcal{D}_\bullet^{\text{un}}(\mu_p)) = \begin{cases} \dim H_{\text{cusp}}^1(\Gamma_1(2; p), \varepsilon_2) & m = 2 \\ \dim H_{\text{cusp}}^1(\Gamma_1(2; p), V_2) & m = 3 \end{cases} \quad (176)$$

Proof. For $m = 2$ this was done before. For $m = 3$ this follows from the above theorem and the Poincaré duality for cuspidal cohomology for $\Gamma_1(3; p)$.

Corollary 2.16: the $m = 2$ case. Thanks to lemma 2.3 and theorem 7.8

$$\dim \mathcal{D}_2(\mu_p) - \dim \Lambda^2 \mathcal{D}_1^{\text{un}}(\mu_p) = 0 - \dim H_{\text{cusp}}^1(\Gamma_1(2; p), \varepsilon(2))$$

One has

$$\dim \mathcal{D}_1^{\text{un}}(\mu_p) = \frac{p-3}{2}; \quad \Rightarrow \quad \dim \Lambda^2 \mathcal{D}_1^{\text{un}}(\mu_p) = \frac{(p-3)(p-5)}{8}$$

Using proposition 1.40 in [Sh] we have, for $p \geq 5$,

$$\dim H^1(X_1(p), \mathbb{Q})^- = \dim H_{\text{cusp}}^1(\Gamma_1(2; p), \varepsilon(2)) = 1 + \frac{p^2-1}{24} - \frac{p-1}{2}$$

(In our case, using the notations of [Sh], $\mu = \frac{p^2-1}{2}$, $\nu_2 = \nu_3 = 0$, $\nu_\infty = p-1$ and $\overline{\Gamma}_1(p) := \Gamma_1(p)/\pm \text{Id}$). Therefore

$$\dim \text{Gr} \mathcal{L}_{-2}^{(l)}(\mu_p) \stackrel{(172)}{=} \dim \mathcal{D}_2(\mu_p) = \frac{(p-5)(p-1)}{12}$$

Corollary 2.16: the $m = 3$ case. We use corollary 7.12 and the following result ([Sh], theorem 2.25)

$$\dim H_{\text{cusp}}^1(\Gamma_1(2; p), V_2 \otimes \varepsilon_2) = 2\left(\frac{p^2-1}{24} - \frac{p-1}{2}\right) + \frac{p-1}{2}$$

Using this and (172) computation of the Euler characteristic gives

$$\begin{aligned} \dim \text{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p) &\stackrel{(172)}{=} \dim \mathcal{D}_3(\mu_p) \\ &= \frac{(p-1)(p-5)(p-3)}{24} - \frac{(p-3)(p-5)(p-7)}{48} - \dim H_{\text{cusp}}^1(\Gamma_1(2;p), V_2 \otimes \varepsilon_2) = \\ &\quad \frac{(p-5)(p^2-2p-11)}{48} \end{aligned}$$

I believe that there exists an explicit degree m polynomial in p giving for $\dim \mathcal{D}_m(\mu_p)$ for all m . There should be a formula for $\chi_m(\widehat{\mathcal{D}}_{\bullet}(\mu_p))$ similar to (176). Notice that there is no closed formula for $\dim H_{\text{cusp}}^2(\Gamma_1(3;p))$.

The computation of $\chi_4(\widehat{\mathcal{D}}_{\bullet}(\mu_p))$ seems to be a feasible problem for the following reasons. First, according to our recent results ([G4]) the relation between the depth m part of the cohomology of the Lie coalgebra $\widehat{\mathcal{D}}_{\bullet}(\mu_p)$ and cohomology of $\Gamma_1(m;p)$ is still in case for $m = 4$:

$$H_{(4)}^i(\widehat{\mathcal{D}}_{\bullet}(\mu_p)) = H^{i+2}(\Gamma_1(4;p), \varepsilon_4); \quad 1 \leq i \leq 4$$

Here ε_4 is the character of $\Gamma_1(4;p)$ given by the determinant. Therefore

$$\chi_4(\widehat{\mathcal{D}}_{\bullet}(\mu_p)) = \sum_{i=3}^6 (-1)^i \dim H^i(\Gamma_1(4;p), \varepsilon_4)$$

Further, the cuspidal cohomology of $\Gamma_1(4;p)$ are in the degrees 4 and 5, and the rest of the cohomology is given by the cohomology at infinity H_{inf}^* , i.e.

$$0 \longrightarrow H_{\text{cusp}}^*(\Gamma_1(4;p), \varepsilon_4) \longrightarrow H^*(\Gamma_1(4;p), \varepsilon_4) \longrightarrow H_{\text{inf}}^*(\Gamma_1(4;p), \varepsilon_4) \longrightarrow 0$$

is an exact sequence. This can be proved using the following facts: besides the trivial representation there is only one unitary cohomological representation of $SL_4(\mathbb{R})$ (the so called special representation) and this representation is tempered (see theorems 5.6 and 6.16 in [VZ] for the general results). Then we employ the arguments similar to the one given in chapter 1 of [LS]. Since the cuspidal cohomology satisfy the Poincaré duality we get

$$\chi_4(\widehat{\mathcal{D}}_{\bullet}(\mu_p)) = \sum_{i=3}^6 (-1)^i \dim H_{\text{inf}}^i(\Gamma_1(4;p), \varepsilon_4)$$

Problem. Compute of the right hand side of this formula.

I would expect that the cuspidal cohomology of $\Gamma_1(3;p)$ will not appear in the answer, so we should have an explicit formula for it.

12. The Galois groups unramified at p . Let us assume that $p \neq l$. Let $\text{Gr}\mathcal{G}_{-m}^{(l)}(\mu_p)^{\text{un}}$ be the maximal quotient of $\text{Gr}\mathcal{G}_{-m}^{(l)}(\mu_p)$ unramified at the place $1 - \zeta_p$. It is easy to see that

$$\text{Gr}\mathcal{G}_{-1}^{(l)}(\mu_p)^{\text{un}} = \text{Hom}\left(\text{the group of the cyclotomic units in } \mathbb{Z}[\zeta_p], \quad \mathbb{Q}_l\right)$$

It follows from theorem 7.8 that

$$\mathrm{Gr}\mathcal{G}_{-2}^{(l)}(\mu_p)^{\mathrm{un}} = \mathrm{Gr}\mathcal{G}_{-2}^{(l)}(\mu_p) \quad \mathrm{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p)^{\mathrm{un}} = \mathrm{Gr}\mathcal{L}_{-3}^{(l)}(\mu_p)$$

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Corrections to the paper [G3]

1. In formulas (10), (35), (36) and formula one line before (35) one needs to replace $S^{w-2}V_2$ by $S^{w-2}V_2 \otimes \varepsilon_2$, where ε_2 is as above.
2. In theorem 6.1 Γ is a finite index subgroup of $GL_2(\mathbb{Z})$; replace $H^{i-1}(\Gamma, V)$ by $H^{i-1}(\Gamma, V \otimes \varepsilon_2)$.
3. Section 4.1, first two lines: use $\mathbb{Z}[X]$ instead of $\mathbb{Z}[[X]]$; line four: $\mathbb{Z}[\mathcal{P}_m]$ instead of $\mathbb{Z}[[\mathcal{P}_m]]$.
4. In theorems 6.2 Γ is a finite index subgroup of $GL_3(\mathbb{Z})$.

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